

2E2 Tutorial Sheet 3 First Term, Solutions¹

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1. Using the Laplace transform solve the differential equation

$$f'' + f' - 6f = e^{-3t} \quad (1)$$

with boundary conditions $f(0) = f'(0) = 0$.

Solution: So, as before, the subsidiary equation is

$$s^2F + sF - 6F = \frac{1}{s+3} \quad (2)$$

or

$$F = \frac{1}{(s+3)^2(s-2)} \quad (3)$$

As before, we do partial fractions

$$\begin{aligned} \frac{1}{(s+3)^2(s-2)} &= \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s-2} \\ 1 &= A(s+3)(s-2) + B(s-2) + C(s+3)^2 \end{aligned} \quad (4)$$

$s = -3$ gives $B = -1/5$ and $s = 2$ gives $C = 1/25$. Putting in $s = 1$ we find

$$1 = -4A + \frac{1}{5} + \frac{16}{25} \quad (5)$$

and so $A = -1/25$. Putting all this together says that

$$f = -\frac{1}{25}e^{-3t} - \frac{t}{5}e^{-3t} + \frac{1}{25}e^{2t} \quad (6)$$

2. Using the Laplace transform solve the differential equation

$$f'' + 6f' + 13f = 0 \quad (7)$$

with boundary conditions $f(0) = 0$ and $f'(0) = 1$. (2)

Solution: So, taking the Laplace transform of the equation we get,

$$s^2F - 1 + 6sF + 13F = 0 \quad (8)$$

and, hence,

$$F = \frac{1}{s^2 + 6s + 13}. \quad (9)$$

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Now, using minus b plus or minus the square root of b squared minus four a c all over two a, we get

$$s^2 + 6s + 13 = 0 \quad (10)$$

if

$$s = \frac{-6 \pm \sqrt{36 - 52}}{2} = -3 \pm 2i \quad (11)$$

which means

$$s^2 + 6s + 13 = (s + 3 - 2i)(s + 3 + 2i) \quad (12)$$

Next, we do the partial fraction expansion,

$$\frac{1}{s^2 + 6s + 13} = \frac{A}{s + 3 - 2i} + \frac{B}{s + 3 + 2i} \quad (13)$$

and multiplying across we get

$$1 = A(s + 3 + 2i) + B(s + 3 - 2i) \quad (14)$$

therefore we choose $s = -3 + 2i$ to get

$$A = \frac{1}{4i} = -\frac{i}{4} \quad (15)$$

and $s = -3 - 2i$ to get

$$B = -\frac{1}{4i} = \frac{i}{4} \quad (16)$$

and so

$$F = -\frac{i}{4} \frac{1}{s + 3 - 2i} + \frac{i}{4} \frac{1}{s + 3 + 2i}. \quad (17)$$

If we take the inverse transform

$$\begin{aligned} f &= -\frac{i}{4} e^{-(3-2i)t} + \frac{i}{4} e^{-(3+2i)t} \\ &= \frac{i}{4} e^{-3t} (e^{-2it} - e^{2it}) \\ &= \frac{1}{2} e^{-3t} \sin 2t \end{aligned} \quad (18)$$

3. Using the Laplace transform solve the differential equation

$$f'' + 6f' + 13f = e^t \quad (19)$$

with boundary conditions $f(0) = 0$ and $f'(0) = 0$. (3)

Solution: Taking the Laplace transform of the equation gives

$$s^2 F + 6sF + 13F = \frac{1}{s - 1} \quad (20)$$

so that

$$F = \frac{1}{(s-1)(s+3+2i)(s+3-2i)}. \quad (21)$$

We write

$$\frac{1}{(s-1)(s+3+2i)(s+3-2i)} = \frac{A}{s+3-2i} + \frac{B}{s+3+2i} + \frac{C}{s-1} \quad (22)$$

giving

$$1 = A(s-1)(s+3+2i) + B(s-1)(s+3-2i) + C(s+3-2i)(s+3+2i). \quad (23)$$

$s = -3 + 2i$ gives

$$1 = A(-4 + 2i)(4i) = A(-8 - 16i) \quad (24)$$

so

$$A = -\frac{1}{8 + 16i} = -\frac{1}{8 + 16i} \frac{8 - 16i}{8 - 16i} = -\frac{1 + 2i}{40} \quad (25)$$

In the same way, $s = -3 - 2i$ leads to

$$B = -\frac{1 - 2i}{40} \quad (26)$$

and, finally, $s = 1$ gives

$$C = \frac{1}{20}. \quad (27)$$

Putting all this together we get

$$F = -\frac{1 + 2i}{40} \frac{1}{s + 3 - 2i} - \frac{1 - 2i}{40} \frac{1}{s + 3 + 2i} + \frac{1}{20} \frac{1}{s - 1} \quad (28)$$

and so

$$\begin{aligned} f &= -\frac{1 + 2i}{40} e^{-(3-2i)t} - \frac{1 - 2i}{40} e^{-(3+2i)t} + \frac{1}{20} e^t \\ &= -\frac{1}{40} e^{-3t} [(1 + 2i)e^{2it} + (1 - 2i)e^{-2it}] + \frac{1}{20} e^t \end{aligned} \quad (29)$$

We then substitute in

$$\begin{aligned} e^{2it} &= \cos 2t + i \sin 2t \\ e^{-2it} &= \cos 2t - i \sin 2t \end{aligned} \quad (30)$$

to end up with

$$f = \frac{1}{20} e^{-3t} [2 \sin 2t - \cos 2t] + \frac{1}{20} e^t \quad (31)$$