The Gauss divergence theorem relates certain three dimensional integrals to surface integral. It states that
\[
\int \int \int_V \text{div} \vec{F} \, dV = \int \int_S \vec{F} \cdot \vec{n} \, dA
\]
where \( V \) is some three dimensional region, \( S \) is its boundary and \( \vec{n} \) is an outward pointing unit normal.

1. (2) Use the Gauss divergence theorem to calculate
\[
\int \int_S \vec{F} \cdot \vec{n} \, dA
\]
where \( \vec{F} = (3x, 3y, 9z) \) and \( S \) is the surface of the cube with unit edges and vertices at \((0,0,0), (1,0,0), (0,0,1), (1,0,1), (0,1,0), (1,1,0), (0,1,1)\) and \((1,1,1)\).

**Solution:**

Recall that \( \text{div} \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \) and, by the quotient rule
\[
\frac{\partial}{\partial x} \left( \frac{x}{r} \right) = \frac{1}{r} - \frac{x^2}{r^3}
\]
Adding up the \( x \), \( y \) and \( z \) contribution gives \( \text{div} \vec{F} = 2/r \).

Now, using the Gauss divergence theorem, calculate
\[
\int \int \int_V \frac{1}{r} \, dV
\]
where \( V \) is the unit sphere of radius one centered on the origin.

**Solution:** First of all, we note from the previous question that

\[
\int \int \int_V \frac{1}{r} \, dV = \int \int \int_V \nabla \cdot \mathbf{F} \, dV \tag{16}
\]

where

\[
\mathbf{F} = \frac{x}{2r} \mathbf{i} + \frac{y}{2r} \mathbf{j} + \frac{z}{2r} \mathbf{k} \tag{17}
\]

and, using the Gauss theorem

\[
\int \int \int_V \nabla \cdot \mathbf{F} \, dV = \int \int \int_S \mathbf{F} \cdot \mathbf{n} \, dA \tag{18}
\]

furthermore,

\[
\mathbf{F} \cdot \mathbf{n} = \frac{x^2}{2r^2} + \frac{y^2}{2r^2} + \frac{z^2}{2r^2} = \frac{1}{2} \tag{19}
\]

and, thus,

\[
\int \int \int_V \frac{1}{r} \, dV = \frac{1}{2} \int \int S \, dA = 2\pi \tag{20}
\]

since the surface area of a sphere \( 4\pi \).