Solns to 2E2 Tutorial Sheet 22[1]

27 April 2004

The Gauss divergence theorem relates certain three dimensional integrals to surface integral. It states that

$$\int\int\int_V \mathrm{div}\,\mathbf{F}\,dV = \int\int_S \mathbf{F}\cdot\mathbf{n}\,dA \tag{1}$$

where V is some three dimensional region, S is its boundary and $\mathbf{n}$ is an outward pointing unit normal.

1. (2) Use the Gauss divergence theorem to calculate

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dA \tag{2}$$

where $\mathbf{F} = (3x, 3y, -9z)$ and S is the surface of the cylinder between the two discs of radius one, perpendicular to the z-axis and with centers at $(0, 0, -1)$ and $(0, 0, 2)$

Solution: Well,

$$\nabla\cdot\mathbf{F} = -3 \tag{3}$$

so

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dA = -3\int\int\int_V dV \tag{4}$$

where V is the cylinder. The cylinder has volume 3π and so the answer is -9π.

Aside It is interesting to do this the other way, one the curved surface of the cylinder the outward pointing normal is

$$\mathbf{n} = \frac{x}{\rho}\mathbf{i} + \frac{y}{\rho}\mathbf{j} \tag{5}$$

where $\rho = \sqrt{x^2 + y^2}$. This means $\mathbf{F}\cdot\mathbf{n} = 3\rho$ and on the curved surface $\rho = 1$ so the curved surface contributes 18π to the total surface integral because the area of the curved surface is 6π. The top has unit normal $\mathbf{k}$ and so $\mathbf{F}\cdot\mathbf{n} = -9z = -18$ on the top. Since the area of the top is 2π the top contributes -36π to the total. As for the bottom, the unit normal is $-\mathbf{k}$ and $z = -1$ on the bottom giving a contribution of 9π. The answer follows from $18\pi - 36\pi + 9\pi = -9\pi$.

2. (2) Use the Gauss divergence theorem to calculate

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dA \tag{6}$$

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where $\mathbf{F} = (3x^2 + z, 3y^2, -9zy)$ and S is the surface of the cube with unit edges and vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$.

Solution: So again, the div$\mathbf{F} = 6x + 6y - 9y = 6x - 3y$ so by Gauss, we want

$$\int_0^1 \int_0^1 \int_0^1 (6x - 3y)\, dxdydz = \int_0^1 \int_0^1 (3x^2 - 3xy]_0^1 dydz = 3\int_0^1 \int_0^1 (3 - 3y) dydz = \frac{3}{2} \tag{7}$$

3. (2) Integrate

$$\int\int_S (y\mathbf{i} + zx\mathbf{j}) \cdot \mathbf{n} dA \tag{8}$$

where S is the surface of a sphere of radius one whose center is the origin.

Solution: Now, here $\mathbf{F} = y\mathbf{i} + zx\mathbf{j}$ and this has zero divergence and hence the integral is zero.

4. (2) What is the outward pointing unit normal to the unit sphere, centered on the origin? *Solution:* So, a sphere or radius one is defined by

$$x^2 + y^2 + z^2 = 1 \tag{9}$$

that is, it is given by $f = 1$ where $f = x^2 + y^2 + z^2$. A normal vector is defined by grad f:

$$\text{grad}\, f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \tag{10}$$

This is clearly ourward pointing, but isn't normal. A normal vector is given by dividing by twice the radius, that is

$$\mathbf{n} = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k} \tag{11}$$

and, for the sphere , $r = 1$ so this is the same as

$$\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{12}$$

What is div $\mathbf{F}$ where $\mathbf{F} = (x/r, y/r, z/r)$?
Solution: Recall that

$$\text{div}\, \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{13}$$

and, by the quotient rule

$$\frac{\partial}{\partial x}\frac{x}{r} = \frac{1}{r} - \frac{x^2}{r^3}. \tag{14}$$

Adding up the x, y and z contribution gives div $\mathbf{F} = 2/r$.
Now, using the Gauss divergence theorem, calculate

$$\int\int\int_V \frac{1}{r}\, dV \tag{15}$$

where V is the unit sphere of radius one centered on the origin.

Solution: First of all, we note from the previous question that

$$\int\int\int_V \frac{1}{r}\,dV = \int\int\int_V \nabla\cdot\mathbf{F}\,dV \tag{16}$$

where

$$F = \frac{x}{2r}\mathbf{i} + \frac{y}{2r}\mathbf{j} + \frac{z}{2r}\mathbf{k} \tag{17}$$

and, using the Gauss theorem

$$\int\int\int_V \nabla\cdot\mathbf{F}\,dV = \int\int_S \mathbf{F}\cdot\mathbf{n}\,dA \tag{18}$$

furthermore,

$$\mathbf{F}\cdot\mathbf{n} = \frac{x^2}{2r^2} + \frac{y^2}{2r^2} + \frac{z^2}{2r^2} = \frac{1}{2} \tag{19}$$

and, thus,

$$\int\int\int_V \frac{1}{r}\,dV = \frac{1}{2}\int\int_S dA = 2\pi \tag{20}$$

since the surface area of a sphere 4π.