2E2 Tutorial Sheet 21 Third Term, Solutions

20 April 2004

1. (1) Find a unit normal to \( z = \sqrt{x^2 + y^2} \) at \((6,8,10)\). Note that this surface is \( f = 0 \) where \( f = z - \sqrt{x^2 + y^2} \).

**Solution:** The normal to \( f = c \) is given by \( \nabla f \), here \( f = z - \sqrt{x^2 + y^2} \) and the surface is \( f = 0 \). Writing \( \rho = \sqrt{x^2 + y^2} \) we have

\[
\nabla f = -\frac{x}{\rho} \hat{i} - \frac{y}{\rho} \hat{j} + \hat{k}
\]

(1)

so, at \((6,8,10)\), \( \rho = 10 \) and the vector, \( \nabla f \), normal to the surface is

\[
\nabla f = -\frac{3}{5} \hat{i} - \frac{4}{5} \hat{j} + \hat{k}
\]

(2)

This has length \( |\nabla f| = \sqrt{2} \) so the unit normal is

\[
\hat{n} = \frac{\nabla f}{|\nabla f|} = -\frac{3}{5\sqrt{2}} \hat{i} - \frac{4}{5\sqrt{2}} \hat{j} + \frac{1}{\sqrt{2}} \hat{k}
\]

(3)

The actual shape of the surface is a cone with its point at the origin.

2. (2) What is the div of

(a) \( v_1(y,z)i + v_2(x,z)j + v_3(x,y)k \)

(b) \( xyz(x\hat{i} + y\hat{j} + z\hat{k}) \)

(c) \( (x^2 + y^2 + z^2)^{-3/2}(x\hat{i} + y\hat{j} + z\hat{k}) \)

**Solution:** Well, for (a), the answer is zero because, in calculating \( \text{div} \), the \( x \) component only gets differentiated by \( y \) and \( z \) and so on. For (b) we have

\[
\nabla \times (xyz, xy^2z, xyz^2) = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
xz & xy^2 & xz^2 \\
y^2 & yz^2 & z^2
\end{vmatrix}
= (xz^2 - y^2z, yz^2 - y^2z, 2yz - xz^2).
\]

(4)

For (c) note that

\[
\frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{3/2} = -3yz
\]

(5)

In other words the numerator is the multiple of the original numerator by the variable we are differentiating by. If you work it out

\[
\frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{3/2} = -3yz
\]

(6)

and this is the same thing and so the curl is zero.

4. (1) Show \( \nabla \times (\nabla f) = 0 \).

**Solution:** As for \( \nabla \times (\nabla f) = 0 \), well

\[
\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}
\]

(7)

so

\[
\nabla \times (\nabla f) = \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \hat{i} + \left( \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \hat{j} + \left( \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \hat{k}
\]

(8)

and again that is zero by changing the order of differentiation.

\[\text{1} \text{ Conor Houghton, houghton@maths.tcd.ie and http://www.maths.tcd.ie/~houghton/ 2E2.html}\]
For completeness, here is the other one, $\nabla \cdot (\nabla \times \mathbf{v}) = 0$. So,

$$\nabla \times \mathbf{v} = \left( \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \mathbf{i} + \left( \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \mathbf{j} + \left( \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \mathbf{k} \tag{12}$$

Now,

$$\nabla \cdot (\nabla \times \mathbf{v}) = \frac{\partial^2 v_3}{\partial y \partial x} - \frac{\partial^2 v_2}{\partial z \partial x} + \frac{\partial^2 v_1}{\partial z \partial y} - \frac{\partial^2 v_3}{\partial y \partial z} + \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_1}{\partial x \partial y} \tag{13}$$

and that is zero if you take into account the fact that the order of differentiation doesn't matter.

5. (2) Show $\text{curl}(f \mathbf{v}) = (\text{grad} \ f) \times \mathbf{v} + f \text{curl} \mathbf{v}$.

Solution: Another one to do by brute force. The vector $f \mathbf{v} = (f v_1, f v_2, f v_3)$ and the curl is

$$\nabla \times (f v_1, f v_2, f v_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f v_1 & f v_2 & f v_3 \end{vmatrix} \tag{14}$$

and so there are two parts corresponding to differentiating the $v_i$, this gives a $f \text{curl} \mathbf{v}$, and differentiating the $f$

$$\nabla \times (f v_1, f v_2, f v_3) = f \nabla \times \mathbf{v} + \left( \frac{\partial f}{\partial y} v_3 - \frac{\partial f}{\partial z} v_2 - \frac{\partial f}{\partial z} v_3 - \frac{\partial f}{\partial y} v_1 \right) \left( \frac{\partial f}{\partial x} v_2 - \frac{\partial f}{\partial z} v_3 - \frac{\partial f}{\partial y} v_1 \right) \tag{15}$$

and, by writing it out, it is easy to see that the second part is $(\text{grad} \ f) \times \mathbf{v}$. 

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