2E2 Tutorial Sheet 21 Third Term, Solutions[1]

20 April 2004

1. (1) Find a unit normal to $z = \sqrt{x^2+y^2}$ at $(6,8,10)$. Note that this surface is $f=0$ where $f = z - \sqrt{x^2+y^2}$

   Solution: The normal to $f = c$ is given by ∇f, here $f = z - \sqrt{x^2+y^2}$ and the surface is $f = 0$. Writing $\rho = \sqrt{x^2+y^2}$ we have

$$\nabla f = -\frac{x}{\rho}\mathbf{i} - \frac{y}{\rho}\mathbf{j} + \mathbf{k} \tag{1}$$

   so, at $(6,8,10)$, $\rho = 10$ and the vector, ∇f, normal to the surface is

$$\nabla f = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} + \mathbf{k} \tag{2}$$

   This has length $|\nabla f| = \sqrt{2}$ so the unit normal is

$$\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|} = -\frac{3}{5\sqrt{2}}\mathbf{i} - \frac{4}{5\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \tag{3}$$

   The actual shape of the surface is a cone with its point at the origin.

2. (2) What is the div of

   (a) $v_1(y,z)\mathbf{i} + v_2(x,z)\mathbf{j} + v_3(x,y)\mathbf{k}$

   (b) $xyz(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

   (c) $(x^2+y^2+z^2)^{-3/2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

   Solution: Well, for (a) the answer is zero becuase in calculating div v_1 is differenciated with respect to x but only depends on y and z, v_2 is differenciated with respect to y but only depends on x and z and v_3 is differenciated with respect to z but only depends on y and z. For (b) we have

$$\nabla \cdot (x^2yz, xy^2z, xyz^2) = 2xyz + 2xyz + 2xyz = 6xyz \tag{4}$$

   where we have used

$$\frac{\partial x^2yz}{\partial x} = 2xyz \tag{5}$$

[1] Conor Houghton, `houghton@maths.tcd.ie` and `http://www.maths.tcd.ie/~houghton/ 2E2.html`

and so on. For (c)

$$\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\frac{2x^2}{(x^2+y^2+z^2)^{5/2}} \tag{6}$$

and if you add up the contibution from v_1, v_2 and v_3 you get zero.

3. (2) What is the curl of

   (a) $v_1(x)\mathbf{i} + v_2(y)\mathbf{j} + v_3(z)\mathbf{k}$

   (b) $xyz(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

   (c) $(x^2+y^2+z^2)^{-3/2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

   Solution: Well, for (a), the answer is zero because, in calculating the curl, the v_1 component only gets differenciated by y and z and so on. For (b) we have

$$\begin{aligned}
\nabla \times (x^2yz, xy^2z, xyz^2) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} \\
&= (xz^2 - xy^2, yx^2 - yz^2, zy^2 - zx^2). \qquad (7)
\end{aligned}$$

   For (c) note that

$$\frac{\partial}{\partial y}\frac{z}{(x^2+y^2+z^2)^{3/2}} = \frac{-3yz}{(x^2+y^2+z^2)^{5/2}} \tag{8}$$

   In other words the numerator is the multiple of the original numerator by the variable we are differenciating by. If you work it out

$$\frac{\partial}{\partial z}\frac{y}{(x^2+y^2+z^2)^{3/2}} = \frac{-3yz}{(x^2+y^2+z^2)^{5/2}} \tag{9}$$

   and this is the same thing and so the curl is zero.

4. (1) Show $\nabla \times (\nabla \mathbf{f}) = 0$.

   Solution: As for $\nabla \times (\nabla \mathbf{f}) = 0$, well

$$\nabla \mathbf{f} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{10}$$

   so

$$\nabla \times (\nabla \mathbf{f}) = \left(\frac{\partial^2 f}{\partial zy} - \frac{\partial^2 f}{\partial yz}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial xz} - \frac{\partial f}{\partial zx}\right)\mathbf{j} + \left(\frac{\partial^2}{\partial yx} - \frac{\partial f}{\partial xy}\right)\mathbf{k} \tag{11}$$

   and again that is zero by changing the order of differenciation.

For completeness, here is the other one, $\nabla \cdot (\nabla \times \mathbf{v}) = 0$. So,

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \mathbf{k} \tag{12}$$

Now,

$$\nabla \cdot (\nabla \times \mathbf{v}) = \frac{\partial^2 v_3}{\partial yx} - \frac{\partial^2 v_2}{\partial xz} + \frac{\partial^2 v_1}{\partial zy} - \frac{\partial^2 v_3}{\partial xy} + \frac{\partial^2 v_2}{\partial xz} - \frac{\partial^2 v_1}{\partial yz} \tag{13}$$

and that is zero if you take into account the fact that the order of differenciation doesn't matter.

5. (2) Show $\text{curl}(f\mathbf{v}) = (\text{grad}\, f) \times \mathbf{v} + f \text{curl}\, \mathbf{v}$.

   Solution: Another one to do by brute force. The vector $f\mathbf{v} = (fv_1, fv_2, fv_3)$ and the curl is

$$\nabla \times (fv_1, fv_2, fv_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fv_1 & fv_2 & fv_3 \end{vmatrix} \tag{14}$$

   and so there are two parts corresponding to differentiating the v_i, this gives a $f\text{curl}\, \mathbf{v}$, and differenciating the f

$$\nabla \times (fv_1, fv_2, fv_3) = f\nabla \times \mathbf{v} + \left(\frac{\partial f}{\partial y} v_3 - \frac{\partial f}{\partial z} v_2, \frac{\partial f}{\partial z} v_1 - \frac{\partial f}{\partial x} v_3, \frac{\partial f}{\partial x} v_2 - \frac{\partial f}{\partial y} v_1 \right) \tag{15}$$

   and, by writing it out, it is easy to see that the second part is $(\text{grad}\, f) \times \mathbf{v}$.