2E2 Tutorial Sheet 20, Solutions[1]

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Vectors calculus. With $f(x, y, z)$ a scalar field the grad of f, ∇f, is

$$\text{grad}\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{1}$$

1. (2) $f = x^2 + 2x^2yz$, find grad f. If $g = x^2yz + 2z$, what is $\nabla f \cdot \nabla g$.

   Solution: So

$$\begin{aligned}\nabla f &= \frac{\partial}{\partial x}\left(x^2 + 2x^2yz\right)\mathbf{i} + \frac{\partial}{\partial y}\left(x^2 + 2x^2yz\right)\mathbf{j} + \frac{\partial}{\partial z}\left(x^2 + 2x^2yz\right)\mathbf{k} \\ &= (2x + 4xyz)\mathbf{i} + 2x^2z\mathbf{j} + 2x^2y\mathbf{k}\end{aligned} \tag{2}$$

   and

$$\begin{aligned}\nabla g &= \frac{\partial}{\partial x}\left(x^2yz + 2z()\right)\mathbf{i} + \frac{\partial}{\partial y}\left(x^2yz + 2z\right)\mathbf{j} + \frac{\partial}{\partial z}\left(x^2yz + 2z\right)\mathbf{k} \\ &= 2xyz\mathbf{i} + x^2z\mathbf{j} + (x^2y + 2)\mathbf{k}\end{aligned} \tag{3}$$

   and

$$\begin{aligned}\nabla f \cdot \nabla g &= 2xyz(2x + 4xyz) + 2x^2zx^2z + 2x^2y(2 + x^2y) \\ &= 4x^2yz + 8x^2y^2z^2 + 2x^4z^2 + 4x^2y + 2x^4y^2\end{aligned} \tag{4}$$

2. (2) Find the directional derivative of $z/(x^2 + y^2)$ in the direction $\mathbf{i}$.

   Solution: Well the directional derivative along $\mathbf{i}$ is grad $f \cdot \mathbf{i}$, in other words the one component of the grad. Furthermore

$$\frac{\partial}{\partial x}\frac{z}{x^2 + y^2} = -\frac{2xz}{(x^2 + y^2)^2} \tag{5}$$

3. (2) $f = xyz$, work out grad f. What is the value of grad f at $(2, 1, 2)$. What is the directional derivative of f in the $\mathbf{i}$-direction at $(2, 1, 2)$.

   Solution: So,

$$\begin{aligned}\text{grad}\, xyz &= \frac{\partial xyz}{\partial x}\mathbf{i} + \frac{\partial xyz}{\partial y}\mathbf{j} + \frac{\partial xyz}{\partial z}\mathbf{k} \\ &= yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}\end{aligned} \tag{6}$$

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At $(2,1,2)$, $x=2$, $y=1$ and $z=2$, so

$$\text{grad}\, f = 2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} \tag{7}$$

The direction derivative in the $\mathbf{i}$ direction is given by $\mathbf{i} \cdot \text{grad}\, f = 2$.

4. (2) $f = x^2 + y^2 + z^2$, find grad f. What is the directional derivative of f in the direction of $\mathbf{b} = (1,1,1)$ at the point $(1,1,1)$. Remember to use a unit vector when working out the directional derivative. What is the directional derivative in the $\mathbf{i}$ direction at $(1,0,0)$; what about the $\mathbf{j}$ direction?

   Solution: So,

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \tag{8}$$

   The directional derivative of f in the direction of $\mathbf{b} = (1,1,1)$ is given by $\widehat{\mathbf{b}} \cdot \nabla f$ where

$$\widehat{\mathbf{b}} = \frac{\mathbf{b}}{|\mathbf{b}|} \tag{9}$$

   is the unit vector in the $\mathbf{b}$ direction. Here $|\mathbf{b}| = \sqrt{3}$ so

$$\widehat{\mathbf{b}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \tag{10}$$

   At $(1,1,1)$, we have $x=y=z=1$ so $\nabla f = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and hence

$$D_{\widehat{\mathbf{b}}} f = \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = 2\sqrt{3} \tag{11}$$

   At $(1,0,0)$ grad $f = 2\mathbf{i}$ so the direction derivative in the $\mathbf{i}$ direction is two and in the $\mathbf{j}$ direction is zero.