2E2 Tutorial Sheet 20, Solutions¹

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Vectors calculus. With f(x, y, z) a scalar field the grad of f, ∇f , is

grad
$$f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$
 (1)

1. (2) $f = x^2 + 2x^2yz$, find grad f. If $g = x^2yz + 2z$, what is $\nabla f \cdot \nabla g$. Solution: So

$$\nabla f = \frac{\partial}{\partial x} \left(x^2 + 2x^2 yz \right) \mathbf{i} + \frac{\partial}{\partial y} \left(x^2 + 2x^2 yz \right) \mathbf{j} + \frac{\partial}{\partial z} \left(x^2 + 2x^2 yz \right) \mathbf{k}$$

= $(2x + 4xyz) \mathbf{i} + 2x^2 z \mathbf{j} + 2x^2 y \mathbf{k}$ (2)

and

$$\nabla g = \frac{\partial}{\partial x} \left(x^2 y z + 2z() \mathbf{i} + \frac{\partial}{\partial y} \left(x^2 y z + 2z \right) \mathbf{j} + \frac{\partial}{\partial z} \left(x^2 y z + 2z \right) \mathbf{k}$$

= $2xyz\mathbf{i} + x^2z\mathbf{j} + (x^2y + 2)\mathbf{k}$ (3)

and

$$\nabla f \cdot \nabla g = 2xyz(2x + 4xyz) + 2x^2zx^2z + 2x^2y(2 + x^2y)$$

= $4x^2yz + 8x^2y^2z^2 + 2x^4z^2 + 4x^2y + 2x^4y^2$ (4)

2. (2) Find the directional derivative of z/(x² + y²) in the direction i.
Solution: Well the directional derivative along i is grad f · i, in other words the one component of the grad. Furthermore

$$\frac{\partial}{\partial x}\frac{z}{x^2 + y^2} = -\frac{2xz}{(x^2 + y^2)^2}$$
(5)

(2) f = xyz, work out grad f. What is the value of grad f at (2,1,2). What is the directional derivative of f in the i-direction at (2,1,2). Solution: So,

grad
$$xyz = \frac{\partial xyz}{\partial x}\mathbf{i} + \frac{\partial xyz}{\partial y}\mathbf{j} + \frac{\partial xyz}{\partial z}\mathbf{k}$$

= $yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ (6)

¹Conor Houghton, houghton@maths.tcd.ie and http://www.maths.tcd.ie/~houghton/ 2E2.html

At (2, 1, 2), x = 2, y = 1 and z = 2, so

$$\operatorname{grad} f = 2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} \tag{7}$$

The direction derivative in the **i** direction is given by $\mathbf{i} \cdot \operatorname{grad} f = 2$.

4. (2) $f = x^2 + y^2 + z^2$, find grad f. What is the directional derivative of f in the direction of $\mathbf{b} = (1, 1, 1)$ at the point (1, 1, 1). Remember to use a unit vector when working out the directional derivative. What is the directional derivative in the \mathbf{i} direction at (1, 0, 0); what about the \mathbf{j} direction?

Solution: So,

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \tag{8}$$

The directional derivative of f in the direction of $\mathbf{b} = (1, 1, 1)$ is given by $\widehat{\mathbf{b}} \cdot \nabla f$ where

$$\widehat{\mathbf{b}} = \frac{\mathbf{b}}{|\mathbf{b}|} \tag{9}$$

is the unit vector in the **b** direction. Here $|\mathbf{b}| = \sqrt{3}$ so

$$\widehat{\mathbf{b}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \tag{10}$$

At (1, 1, 1), we have x = y = z = 1 so $\nabla f = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and hence

$$D_{\hat{\mathbf{b}}}f = \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = 2\sqrt{3} \tag{11}$$

At (1, 0, 0) grad $f = 2\mathbf{i}$ so the direction derivative in the \mathbf{i} direction is two and in the \mathbf{j} direction is zero.