1. Using the Laplace transform solve the differential equation

\[ f'' - 4f' + 3f = 1 \]  
(1)

with boundary conditions \( f(0) = f'(0) = 0 \).

**Solution:** First, take the Laplace transform of the equation. Since \( f'(0) = f(0) = 0 \), if \( \mathcal{L}(f) = F(s) \) then \( \mathcal{L}(f') = sF(s) \) and \( \mathcal{L}(f'') = s^2F(s) \). Thus, the subsidiary equation is

\[ s^2F - 4sF + 3F = \frac{1}{s} \]  
(2)

and so

\[
(s^2 - 4s + 3)F = \frac{1}{s} \\
F = \frac{1}{s} \cdot \frac{1}{s^2 - 4s + 3} 
\]  
(3)

and, since \( s^2 - 4s + 3 = (s - 3)(s - 1) \), this gives

\[ F = \frac{1}{s(s - 3)(s - 1)} \]  
(4)

Before we can invert this, we need to do a partial fraction expansion.

\[
\frac{1}{s(s - 3)(s - 1)} = \frac{A}{s} + \frac{B}{s - 3} + \frac{C}{s - 1} \\
1 = A(s - 3)(s - 1) + Bs(s - 1) + Cs(s - 3) 
\]  
(5)

So substituting in \( s = 0 \) we get \( A = 1/3 \), \( s = 3 \) gives \( B = 1/6 \) and \( s = 1 \) gives \( C = -1/2 \). Hence

\[ F = \frac{1}{3s} + \frac{1}{6(s - 3)} - \frac{1}{2(s - 1)} \]  
(6)

and so

\[ f(t) = \frac{1}{3} + \frac{1}{6}e^{3t} - \frac{1}{2}e^t \]  
(7)
2. Using the Laplace transform solve the differential equation

\[ f'' - 4f' + 3f = 2e^t \]  \hspace{1cm} (8)

with boundary conditions \( f(0) = f''(0) = 0 \).

**Solution:** This time we have \( \mathcal{L}(2e^t) = 2/(s - 1) \) on the right hand side. This means that the subsidiary equation is

\[ (s^2 - 4s + 3)F = \frac{2}{s - 1} \]  \hspace{1cm} (9)

so

\[ F = \frac{2}{(s - 1)^2(s - 3)} \]  \hspace{1cm} (10)

We need to do partial fractions again, but this is one of those cases with a repeated root:

\[ \frac{1}{(s - 1)^2(s - 3)} = \frac{A}{s - 1} + \frac{B}{(s - 1)^2} + \frac{C}{s - 3} \]  \hspace{1cm} (11)

and multiplying across

\[ 1 = A(s - 1)(s - 3) + B(s - 3) + C(s - 1)^2 \]  \hspace{1cm} (12)

so \( s = 1 \) gives \( B = -1/2 \) and \( s = 3 \) gives \( C = 1/4 \). No value of \( s \) gives \( A \) on its own, so we try \( s = 2 \):

\[ 1 = -A + \frac{1}{2} + \frac{1}{4} \]  \hspace{1cm} (13)

which means that \( A = -1/4 \). Hence

\[ F = -\frac{1}{2(s - 1)} - \frac{1}{(s - 1)^2} + \frac{1}{2(s - 3)} \]  \hspace{1cm} (14)

and

\[ f = -\frac{1}{2}e^t - te^t + \frac{1}{2}e^{3t} \]  \hspace{1cm} (15)

3. Using the Laplace transform solve the differential equation

\[ f'' - 4f' + 3f = 0 \]  \hspace{1cm} (16)

with boundary conditions \( f(0) = 1 \) and \( f'(0) = 1 \).

**Solution:** In this example there are non-zero boundary conditions. Since

\[ \mathcal{L}(f') = sF - f(0) \]  \hspace{1cm} (17)
\[ \mathcal{L}(f'') = s^2F - sf(0) - f'(0) \]  \hspace{1cm} (18)

the subsidiary equation in this case is

\[ s^2F - s - 1 - 4sF + 4 + 3F = 0 \]  \hspace{1cm} (19)

\[ 2 \]
so
\[(s^2 - 4s + 3)F = s - 3.\] \hspace{1cm} (20)

Hence
\[F = \frac{1}{s - 1}\] \hspace{1cm} (21)

and
\[f(t) = e^t\] \hspace{1cm} (22)

4. Using the Laplace transform solve the differential equation
\[y'' - 2ay' + a^2y = 0\] \hspace{1cm} (23)

with boundary conditions \(y'(0) = 1\) and \(y(0) = 0\). \(a\) is some real constant.

\textit{Solution:} Taking the Laplace transform we get
\[s^2Y - 1 - 2aY + a^2Y = 0\] \hspace{1cm} (24)

and hence
\[Y = \frac{1}{(s - a)^2}\] \hspace{1cm} (25)

which means that
\[y = te^{at}\] \hspace{1cm} (26)