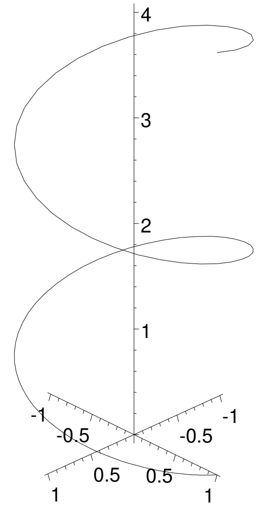
2E2 Tutorial Sheet 19 Third Term, Solutions[1]

6 April 2004

1. (2) $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}-9\mathbf{k}$. find $\mathbf{a}+\mathbf{b}$, $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$. $\mathbf{c} = (0,7,2)$, find $\mathbf{a}\cdot\mathbf{c}$ and $\mathbf{b}\times\mathbf{c}$.

   Solution: The answers are $\mathbf{a}+\mathbf{b} = 5\mathbf{i}+2\mathbf{j}-6\mathbf{k}$, $\mathbf{a}\cdot\mathbf{b} = -21$, $\mathbf{a}\times\mathbf{b} - 18\mathbf{i}+33\mathbf{j}-4\mathbf{k}$, $\mathbf{a}\cdot\mathbf{c} = 20$ and $\mathbf{b}\times\mathbf{c} = (-63,-4,14) = -63\mathbf{i}-4\mathbf{j}+14\mathbf{k}$.



2. (3) $\mathbf{r}(t) = \sin\pi t^2\mathbf{i}+\cos\pi t^2\mathbf{j}+t^2\mathbf{k}$ for $t\geq 0$ is a curve in space. Work out its length between its starting point, $(0,1,0)$, and $(0,-1,1)$.

   Solution: First of all note that $\mathbf{r}(0) = \mathbf{j}$ and $\mathbf{r}(1) = -\mathbf{j}+\mathbf{k}$ and so we are interested in the length of the curve from $t=0$ to $t=1$. We use the formula for the length of a curve from t_1 to t_2,

$$l = \int_{t_1}^{t_2}\sqrt{\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}}dt \tag{1}$$

   In this case

$$\frac{d\mathbf{r}}{dt} = 2t\pi\cos\pi t^2\mathbf{i}+2t\pi\sin\pi t^2\mathbf{j}+2t\mathbf{k} \tag{2}$$

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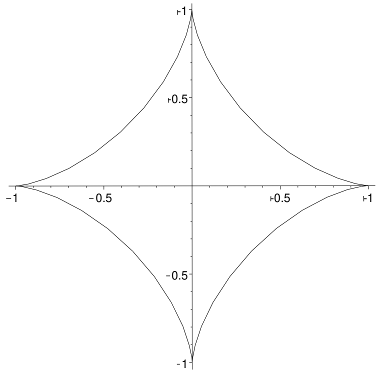
which means

$$l = \int_0^1 2t\sqrt{1+\pi^2}dt = \sqrt{1+\pi^2}\,t^2\big]_0^1 = \sqrt{1+\pi^2} \tag{3}$$

Aside An actual picture of the curve is given in Figure 1. Notice that the parameter t plays a hidden role, it tells you how fast $\mathbf{r}(t)$ moves along the curve as t changes. The curve $\mathbf{s}(t) = \sin\pi t\mathbf{i}+\cos\pi t\mathbf{j}+t\mathbf{k}$ is the same as a curve in space, but different values of t correspond to different points. For example $\mathbf{s}(\sqrt{2}) = \mathbf{j}+2\mathbf{k}$ so the point $(0,1,2)$ is on the curve and corresponds to $t=\sqrt{2}$. However, $\mathbf{s}$is the same curve and the point $(0,1,2)$ is on it, but corresponds to $t=2$.



3. The figure shows the curve $\mathbf{r}(t) = \cos^3 t\mathbf{i}+\sin^3 t\mathbf{j}$. Work out its total length.

   Solution: So, here the curve is closed and so we need to decide a starting point. For $t=0$, $\mathbf{r}=\mathbf{i}$ and if $\mathbf{r}(t)=\mathbf{i}$ this implies that $t=0$ or 2π and so on. Thus, we can find the total length by integrating from $t_1=0$ and $t_2=2\pi$. Now, by differenciating

$$\mathbf{r}'(t) = 3\sin t\cos^2 t\mathbf{i}+2\cos t\sin^2 t\mathbf{j} \tag{4}$$

   and so $|\mathbf{r}'| = \pm 3\sqrt{(\sin t\cos^2 t)^2+(\cos t\sin^2 t)^2} = 3|\sin t\cos t|$. The slightly confusing point is that the length of $\mathbf{r}'$ is always positive so you have to take the positive or negative square root according to which gives a positive answer. Now, using the periodicity of the sine

$$\frac{3}{2}\int_0^{2\pi}|\sin 2t|dt = 6\int_0^{\pi/2}\sin 2tdt \tag{5}$$

   and so the length is

$$6\int_0^{\pi/2}\sin 2tdt = 3\cos 2t]_0^{\pi/2} = 6 \tag{6}$$