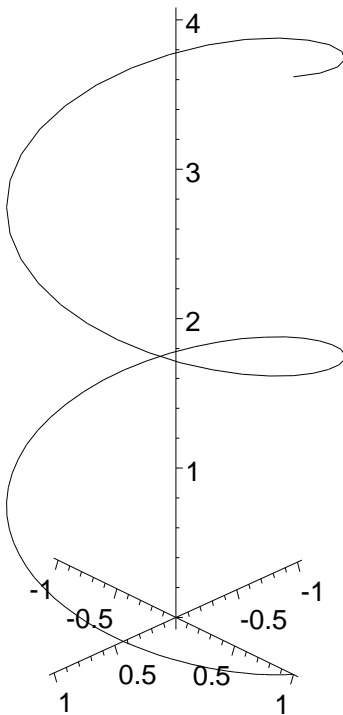


2E2 Tutorial Sheet 19 Third Term, Solutions¹

6 April 2004

1. (2) $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 9\mathbf{k}$. find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$. $\mathbf{c} = (0, 7, 2)$, find $\mathbf{a} \cdot \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$.

Solution: The answers are $\mathbf{a} + \mathbf{b} = 5\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$, $\mathbf{a} \cdot \mathbf{b} = -21$, $\mathbf{a} \times \mathbf{b} = 18\mathbf{i} + 33\mathbf{j} - 4\mathbf{k}$, $\mathbf{a} \cdot \mathbf{c} = 20$ and $\mathbf{b} \times \mathbf{c} = (-63, -4, 14) = -63\mathbf{i} - 4\mathbf{j} + 14\mathbf{k}$.



2. (3) $\mathbf{r}(t) = \sin \pi t^2 \mathbf{i} + \cos \pi t^2 \mathbf{j} + t^2 \mathbf{k}$ for $t \geq 0$ is a curve in space. Work out its length between its starting point, $(0, 1, 0)$, and $(0, -1, 1)$.

Solution: First of all note that $\mathbf{r}(0) = \mathbf{j}$ and $\mathbf{r}(1) = -\mathbf{j} + \mathbf{k}$ and so we are interested in the length of the curve from $t = 0$ to $t = 1$. We use the formula for the length of a curve from t_1 to t_2 ,

$$l = \int_{t_1}^{t_2} \sqrt{\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}} dt \quad (1)$$

In this case

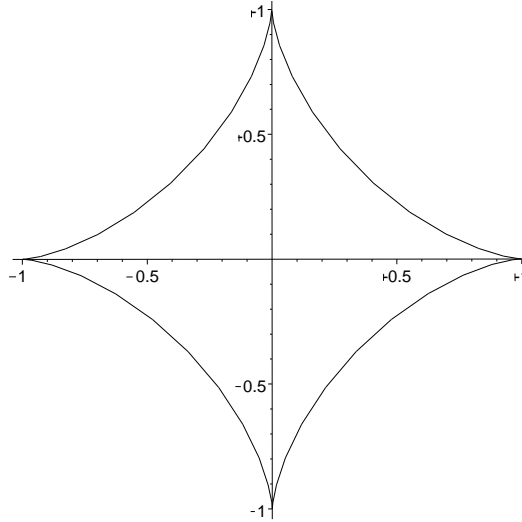
$$\frac{d\mathbf{r}}{dt} = 2t\pi \cos \pi t^2 \mathbf{i} + 2t\pi \sin \pi t^2 \mathbf{j} + 2t\mathbf{k} \quad (2)$$

¹Conor Houghton, houghton@maths.tcd.ie and <http://www.maths.tcd.ie/~houghton/2E2.html>

which means

$$l = \int_0^1 2t\sqrt{1+\pi^2}dt = \sqrt{1+\pi^2} t^2 \Big|_0^1 = \sqrt{1+\pi^2} \quad (3)$$

Aside An actual picture of the curve is given in Figure 1. Notice that the parameter t plays a hidden role, it tells you how fast $\mathbf{r}(t)$ moves along the curve as t changes. The curve $\mathbf{s}(t) = \sin \pi t \mathbf{i} + \cos \pi t \mathbf{j} + t \mathbf{k}$ is the same as a curve in space, but different values of t correspond to different points. For example $\mathbf{s}(\sqrt{2}) = \mathbf{j} + 2\mathbf{k}$ so the point $(0, 1, 2)$ is on the curve and corresponds to $t = \sqrt{2}$. However, \mathbf{s} is the same curve and the point $(0, 1, 2)$ is on it, but corresponds to $t = 2$.



3. The figure shows the curve $\mathbf{r}(t) = \cos^3 t \mathbf{i} + \sin^3 t \mathbf{j}$. Work out its total length.

Solution: So, here the curve is closed and so we need to decide a starting point. For $t = 0$, $\mathbf{r} = \mathbf{i}$ and if $\mathbf{r}(t) = \mathbf{i}$ this implies that $t = 0$ or 2π and so on. Thus, we can find the total length by integrating from $t_1 = 0$ and $t_2 = 2\pi$. Now, by differentiating

$$\mathbf{r}'(t) = 3 \sin t \cos^2 t \mathbf{i} + 2 \cos t \sin^2 t \mathbf{j} \quad (4)$$

and so $|\mathbf{r}'| = \pm 3\sqrt{(\sin t \cos^2 t)^2 + (\cos t \sin^2 t)^2} = 3|\sin t \cos t|$. The slightly confusing point is that the length of \mathbf{r}' is always positive so you have to take the positive or negative square root according to which gives a positive answer. Now, using the periodicity of the sine

$$\frac{3}{2} \int_0^{2\pi} |\sin 2t| dt = 6 \int_0^{\pi/2} \sin 2t dt \quad (5)$$

and so the length is

$$6 \int_0^{\pi/2} \sin 2t dt = 3 \cos 2t \Big|_0^{\pi/2} = 6 \quad (6)$$