2E2 Tutorial Sheet 18 Third Term, Solutions[1]

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Series solution question: (3) The Legendre equation is

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0 \tag{1}$$

and this has series solution with recursion relation

$$a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+2)(n+1)}a_n \tag{2}$$

If l is an integer one of the two series terminates to give a polynomial. This polynomial is called $P_l(x)$ normalized by requiring $P_l(1) = 1$. Write down $P_1(x)$ and $P_3(x)$. *Solution:* Well writing down the first Legendre polynomials is not too hard since the recursion relation is given. For $l = 1$ the recursion relation becomes

$$a_{n+2} = -\frac{(1-n)(1+n+1)}{(n+2)(n+1)}a_n \tag{3}$$

so the odd series terminates after a_1 because with $n = 1$ the $(1-n)$ is zero. Thus the polynomial is

$$P_1(x) = a_1 x \tag{4}$$

and so $1 = P_1(1) = a_1$ means $a_1 = 1$ and

$$P_1(x) = x. \tag{5}$$

For $l = 3$ the recursion relation becomes

$$a_{n+2} = -\frac{(3-n)(3+n+1)}{(n+2)(n+1)}a_n \tag{6}$$

so the odd series terminates after a_3 and

$$a_3 = -\frac{(3-1)(3+1+1)}{(1+2)(1+1)}a_1 = -\frac{10}{6}a_1. \tag{7}$$

Thus,

$$P_3(x) = a_1\left(x - \frac{10}{6}x^3\right) \tag{8}$$

and $P_3(1) = 1$ implies $1 = -(2/3)a_1$ and hence

$$P_3(x) = -\frac{1}{2}(3x - 5x^3) \tag{9}$$

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Revision of vectors. The idea in the rest of the sheet is to revise adding vectors, finding their dot products, finding their cross products and working out the scalar triple product.

Adding vectors. Vector are added component by component so if $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ then $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

Length. The length of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^3}$

Dot product. The dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is $|\mathbf{u}||\mathbf{v}|\cos\theta$ where θ is the angle between them. In terms of components $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$

Cross product. The cross product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is a vector with length $|\mathbf{u}||\mathbf{v}|\sin\theta$ where θ is the angle between them. It points perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. In terms of components

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the three basis vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.

Scalar triple product of three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

1. (5) For

$$\mathbf{a} = (1, 2, 0), \quad \mathbf{b} = (-3, 2, 0), \quad \mathbf{c} = (2, 3, 4), \quad \mathbf{d} = (6, -7, 2).$$

calculate (i) $\mathbf{a} + \mathbf{b}$, (ii) $\mathbf{a} \cdot \mathbf{b}$, (iii) $|\mathbf{a}|$, (iv) $\mathbf{a} \times \mathbf{b}$, (v) $\mathbf{b} \times \mathbf{a}$, (vi) $\mathbf{b} \times \mathbf{c}$, (vii) $|\mathbf{a} \times \mathbf{c}|$, (viii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. (ix) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, (x) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

Solution: (i) Vectors add component by component so $\mathbf{a} + \mathbf{b} = (1 - 3, 2 + 2, 0 + 0) = (-2, 4, 0)$.
(ii) The dot product sums the multiples of the components, $\mathbf{a} \cdot \mathbf{b} = 1 \times (-3) + 2 \times 2 + 0 \times 0 = 1$.
(iii) $\mathbf{a} = \sqrt{1 + 2^2} = \sqrt{5}$.
(iv) And here we use the formula

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ -3 & 2 & 0 \end{vmatrix} \\ &= (2 \times 0 - 0 \times 2)\mathbf{i} - (1 \times 0 - 0 \times (-3))\mathbf{j} + (1 \times 2 - 2 \times (-3))\mathbf{k} \\ &= 8\mathbf{k} \end{aligned}$$

(v) Well if you multiply this one out, it is minus the previous one: the cross product is anti-commutative. the answer is $-8\mathbf{k}$.
(vi)

$$\begin{aligned} \mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 0 \\ 2 & 3 & 4 \end{vmatrix} \\ &= (2 \times 4 - 0 \times 3)\mathbf{i} - (-3 \times 4 - 0 \times 2)\mathbf{j} + (-3 \times 3 - 2 \times 3)\mathbf{k} \\ &= 8\mathbf{i} + 12\mathbf{j} - 13\mathbf{k} \end{aligned}$$

(vii) There are two ways to do this one, we can work out the cross product and then calculate its length or we can use the formula

$$|\mathbf{a}\times\mathbf{b}| = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2}$$

which is given in (K 422 P38). So, by direct calculation

$$\begin{aligned}
\mathbf{a}\times\mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 2 & 3 & 4 \end{vmatrix} \\
&= (2\times 4 - 0\times 3)\mathbf{i} - (-3\times 4 - 0\times 2)\mathbf{j} + (-3\times 3 - 2\times 3)\mathbf{k} \\
&= 8\mathbf{i} - 4\mathbf{j} - \mathbf{k}
\end{aligned}$$

and so $|\mathbf{b}\times\mathbf{c}| = \sqrt{64+16+1} = \sqrt{81} = 9$. The other way has $|\mathbf{a}|^2 = 5$ and $|\mathbf{c}|^2 = 29$ whereas $\mathbf{a}\cdot\mathbf{c} = 8$ and hence

$$|\mathbf{a}\times\mathbf{c}| = \sqrt{145-64} = 9$$

(vii) The scalar triple products are best done using the formula

$$\mathbf{w}.(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

In this case, this gives

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 2 & 0 \\ -3 & 2 & 0 \\ 2 & 3 & 4 \end{vmatrix} = 32$$

(ix) Well we know $\mathbf{a}\cdot\mathbf{b} = 1$ so $(\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \mathbf{c}$.
(x) You can work this one out directly, or

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = 32$$

by the cyclic property of the trace which will be demonstrated in the next problem sheet.