2E2 Tutorial Sheet 17 Second Term[1]

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1. (3) Assuming the solution of
$$y'' - 3x^2y = 0 \tag{1}$$
has a series expansion about $x = 0$ work out the recursion relation and write out the first four non-zero terms if $y(0) = 1$ and $y'(0) = 1$.

We substitute
$$y = \sum_{n=0}^{\infty} a_n x^n \tag{2}$$
into the equation. This gives
$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} 3a_n x^{n+2} = 0 \tag{3}$$
The problem here is with the powers of x. The easiest thing is to change everything to the highest power, in this case $n + 2$. Hence, put $m + 2 = n - 2$ in the first sum
$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=-4}^{\infty} (m+4)(m+3)a_{m+4}x^{m+2}. \tag{4}$$
and substitute that back into the equation, writing m as n:
$$\sum_{n=-4}^{\infty} (n+4)(n+3)a_{n+4}x^{n+2} - \sum_{n=0}^{\infty} 3a_n x^{n+2} = 0 \tag{5}$$
and so the problem now is that the ranges are different. We need to take out the first few term of the first sum, well, the $n = -4$ and $n = -3$ terms are zero and so
$$\sum_{n=-4}^{\infty} (n+4)(n+3)a_{n+4}x^{n+2} = 2a_2 + 6a_3x + \sum_{n=0}^{\infty} (n+4)(n+3)a_{n+4}x^{n+2}. \tag{6}$$
Now the equation reads
$$2a_2 + 6a_3x + \sum_{n=0}^{\infty} (n+4)(n+3)a_{n+4}x^{n+2} - \sum_{n=0}^{\infty} 3a_n x^{n+2} = 0 \tag{7}$$
or
$$2a_2 + 6a_3x + \sum_{n=0}^{\infty} [(n+4)(n+3)a_{n+4} - 3a_n]\, x^{n+2} = 0. \tag{8}$$

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Thus
$$\begin{aligned} a_2 &= 0 \\ a_3 &= 0 \\ a_{n+4} &= \frac{3}{(n+4)(n+3)}a_n \end{aligned} \tag{9}$$
where the recursion relation applies for $n = 0, 1, \ldots$. Now, $y(0) = 1$ implies $a_0 = 1$ and $y'(0) = 1$ implies $a_1 = 1$, next, with $n = 0$, t he recursion gives
$$a_4 = \frac{1}{4}a_0 = \frac{1}{4} \tag{10}$$
and with $n = 1$
$$a_5 = \frac{3}{20}a_1 = \frac{3}{20}. \tag{11}$$
Now since $a_2 = a_3 = 0$ the $n = 2$ recursion gives $a_6 = 0$ and the $n = 3$ recursion g ives $a_7 = 0$. However, $n = 4$ gives
$$a_8 = \frac{3}{32}a_4 = \frac{3}{128} \tag{12}$$
and so
$$y = 1 + x + \frac{1}{4}x^4 + \frac{3}{20}x^5 + \frac{3}{128}x^8 + \ldots. \tag{13}$$

Aside. In the above we made all the powers the same as the highest power, this is usually the easiest thing, but it is just a matter of convenience. If we had decided to make them equal the smallest power i nstead, we would have substituted $n+2 = m-2$ in the second sum to get
$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=4}^{\infty} 3a_{n-4}x^{n-2} = 0 \tag{14}$$
and we would then remove the first four term from the first sum to get
$$2a_2 + 6a_3x + \sum_{n=4}^{\infty} \left[n(n-1)a_n x^{n-2} - 3a_{n-4}\right] x^{n-2} = 0 \tag{15}$$
and so
$$\begin{aligned} a_2 &= 0 \\ a_3 &= 0 \\ a_n &= \frac{3}{n(n-1)}a_{n-4} \end{aligned} \tag{16}$$

where now the recursion relation applies to $n = 4, 5, \dots$ because that is what is in the sum. Another way of proceeding is to define $a_{-4} = a_{-3} = a_{-2} = a_{-1} = 0$ and then rewrite the equation as

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} 3a_{n-4}x^{n-2} = 0 \tag{17}$$

and carry on from there.

2. (2) Assuming the solution of

$$y' - 3xy = 2 \tag{18}$$

has a series expansion about $x = 0$, work out the recursion relation and write out the first four non-zero terms.

Solution: The complication here is that unlike the other examples we have examined, this equation is an inhomogeneous equation. However,the thing to do is to press on with the same methods and hope for the best.

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{19}$$

gives, when substituted into the equation,

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 3a_n x^{n+1} = 2 \tag{20}$$

and so the first problem is with the powers of x, let $m + 1 = n - 1$ in the first sum to give

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{m=-2}^{\infty} (m+2)a_{m+2}x^{m+1} \tag{21}$$

and, noting the the $m = -2$ term is zero, we take out the first two terms to get

$$a_1 + \sum_{n=0}^{\infty} [(n+2)a_{n+2} - 3a_n]x^{n+1} = 2. \tag{22}$$

Now, notice that the summand starts with an x term and so we get

$$\begin{aligned} a_1 &= 2 \\ a_{n+2} &= \frac{3}{n+2}a_n. \end{aligned} \tag{23}$$

Thus,

$$a_2 = \frac{3}{2}a_0 \tag{24}$$

and

$$a_3 = a_1 = 2. \tag{25}$$

Hence

$$y = a_0\left(1 + \frac{3}{2}x^2 + \dots\right) + 2x + 2x^3 + \dots \tag{26}$$

and we see that the solution to this inhomogeneous solution has the usual structure: a particular part and a solution to the homogeneous equation depending on an arbitrary constant.

3. (3) Use the method of Froebenius to find series solutions for

$$xy'' + 2y' + xy = 0 \tag{27}$$

about $x = 0$.

Solution: So, since we are told to use the method of Froebenius, we substitute

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{28}$$

Even if you weren't told this was a method of Froebenius problem, you would soon find that the ordinary method doesn't give two solutions. Alternatively, you could notice that if you write the equation so nothing multiplies y'' you have coefficients with singularities, that is in this case, the $2/x$ multiplying y'.

Now, substituting into the equation gives

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 2(n+r)]a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0. \tag{29}$$

so, moving the first power up to the second one, this gives

$$\sum_{n=-2}^{\infty} [(n+2+r)(n+r+1) + 2(n+r+2)]a_{n+2}x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0 \tag{30}$$

or, taking the first two terms out

$$\begin{aligned} r(r+1)a_0x^{r-1} + (r+1)(r+2)a_1x^r \quad &+ \\ \sum_{n=0}^{\infty}[(n+2+r)(n+r+3)]a_{n+2}x^{n+r+1} \quad &+ \quad \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0. \end{aligned} \tag{31}$$

So, if $r = 0$ or $r = -1$ then there is no constraint on a_0. Notice that $r = -1$ allows two solutions because, if $r = -1$ there is no equation for either a_0 or a_1. For $r = -1$ the recursion is

$$a_{n+2} = \frac{1}{(n+1)(n+2)} \tag{32}$$

so the first few non-zero terms are

$$y = \frac{1}{x}\left[a_0\left(1+\frac{1}{2}x^2+\frac{1}{24}x^4+\dots\right)+a_1\left(x+\frac{1}{6}x^3\dots\right)\right] \tag{33}$$

For $r=0$ the recursion is

$$a_{n+2} = \frac{1}{(n+2)(n+3)} \tag{34}$$

and $a_1 = 0$, this means that the $r=0$ solution is

$$y = a_0\left(1+\frac{1}{6}x^2+\dots\right) \tag{35}$$

Notice that the $r=0$ solution is actually just the a_1 solution for $r=-1$. This is just as well because there would be too many solutions otherwise. Notice the subtle way the method of Froebenius problems often work out. There is quite a lot to this subject we have only touched on. As an aside, notice the the solutions to the differential are $\cos x/x$ and $\sin x/x$. WWriting these out as series will give the same thing as above.