2E2 Tutorial Sheet 16 Second Term, Solutions[1]

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1. (3) Assuming the solution of

$$(1-x)y' + y = 0 \tag{1}$$

has a series expansion about $x = 0$ work out the recursion relation. Write out the first few terms and show that the series terminates to give $y = A(1-x)$ for arbitrary A.

Solution: So we begin by writing

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

and so by differentiation we get

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} \tag{3}$$

and hence

$$xy' = \sum_{n=0}^{\infty} a_n n x^n. \tag{4}$$

Thus, substituting the differential equation we get

$$\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \tag{5}$$

In order to make progress we need to rewrite the first of these three series so that it is in the form

$$\sum_{n=0}^{\infty} \text{stuff}_n x^n \tag{6}$$

so that all three bits in the equation match. Well, let $m = n - 1$ in the expression for y', (3), to get

$$y' = \sum_{m=0}^{\infty} a_{m+1}(m+1)x^m. \tag{7}$$

In fact, this looks at first like it gives

$$y' = \sum_{m=-1}^{\infty} a_{m+1}(m+1)x^m \tag{8}$$

[1]Conor Houghton, `houghton@maths.tcd.ie` and `http://www.maths.tcd.ie/~houghton/ 2E2.html`

but the $m = -1$ term is zero, so that's fine. Now m is just an index so we ca rename it n, don't get confused, this isn't the original n, we just want all parts o the equation to look the same.

In fact, we now have

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \quad (9$$

and we can group this all together to give

$$\sum_{n=0}^{\infty} [a_{n+1}(n+1) + (1-n)a_n]x^n = 0. \quad (10$$

The recursion relation is

$$a_{n+1} = -\left(\frac{1-n}{1+n}\right) a_n \quad (11$$

and this applies to n from zero upwards since that is what appears in the sum sign

Starting at $n = 0$ we have

$$a_1 = -a_0. \quad (12$$

For $n = 1$ we get

$$a_2 = 0 \quad (13$$

and the series terminates here because every term is something multiplied by th one before and so if a_2 is zero the rest of the series is zero. Thus $y = a_0(1-x)$ fo arbitrary a_0.

2. (3) Assuming the solution of

$$(1-x^2)y' - 2xy = 0 \quad (14$$

has a series expansion about $x = 0$, work out the recursion relation and write ou the first four non-zero terms.

Solution: Assuming the solution of

$$(1-x^2)y' - 2xy = 0 \quad (15$$

has a series expansion about $x = 0$, work out the recursion relation and write ou the first four non-zero terms.

Answer: Once again let

$$y = \sum_{n=0}^{\infty} a_n x^n \quad (16$$

and, as before,

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} \tag{17}$$

so

$$x^2 y' = \sum_{n=0}^{\infty} a_n n x^{n+1} \tag{18}$$

and finally

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1}. \tag{19}$$

The equation then reads

$$\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1}. \tag{20}$$

Once again, the first term is a problem because it doesn't have the same form as the other two. So, take

$$\sum_{n=0}^{\infty} a_n n x^{n-1} \tag{21}$$

and put $n-1 = m+1$ and, hence, $n = m+2$. When $n = 0$, $m = -2$ and when $n = 1$, $m = -1$. Thus

$$\sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{m=-2}^{\infty} a_{m+2}(m+2)x^{m+1} \tag{22}$$

and, once again renaming m as n we get

$$\sum_{n=-2}^{\infty} (n+2)a_{n+2}x^{n+1} - \sum_{n=0}^{\infty} n a_n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0. \tag{23}$$

The problem now is with the range that the first sum runs over. The $n = -2$ term is no problem, it is zero, but the $n = -1$ term is a_1. Thus, we write

$$\sum_{n=-2}^{\infty} (n+2)a_{n+2}x^{n+1} = a_1 + \sum_{n=0}^{\infty} (n+2)a_{n+2}x^{n+1} \tag{24}$$

and the equation becomes

$$a_1 + \sum_{n=0}^{\infty} (n+2)a_{n+2}x^{n+1} - \sum_{n=0}^{\infty} a_n n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0. \tag{25}$$

Thus

$$a_1 + \sum_{n=0}^{\infty} [(n+2)a_{n+2} - n a_n - 2a_n] x^{n+1} = 0. \tag{26}$$

Notice that the summand starts with the x term. The recursion relation is therefor

$$a_{n+2} = a_n \tag{27}$$

with the additional conditions $a_1 = 0$. Hence, $a_6 = a_4 = a_2 = a_0$, $a_5 = a_3 = a_1 =$ and so on. The first four nonzero terms of the expansion gives

$$y = a_0(1 + x^2 + x^4 + x^6 + \ldots). \tag{28}$$

3. (2) Assuming the solution of

$$y'' - 3y' + 2y = 0 \tag{29}$$

has a series expansion about $x = 0$, by substitution, work out the recursion relatior If $y(0) = 1$ and $y'(0) = 0$ what are the first three non-zero terms.

Solution: Again

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{30}$$

so

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \tag{31}$$

and

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \tag{32}$$

Thus,

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 3\sum_{n=0}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0 \tag{33}$$

Again, we want to make each part look the same. As before, changing the index g ves

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n. \tag{34}$$

The same thing can be done with the y'': let $m = n-2$ to get

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=-2}^{\infty} (m+1)(m+2)a_{m+2}x^m \tag{35}$$

and the $m = -2$ and $m = -1$ terms are both zero, so, renaming the m as n we get

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n - 3\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0 \tag{36}$$

and this gives

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n]x^n = 0. \tag{37}$$

The recursion relation is

$$(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0. \tag{38}$$

Now apply the initial conditions, $y(0) = 1$ implies that $a_0 = 1$, $y'(0) = 0$ implies $a_1 = 0$. For $n = 0$ the recursion relation gives

$$2a_2 - 3a_1 + 2a_0 = 0 \tag{39}$$

and so $a_2 = -a_0 = -1$. Next $n = 1$ gives

$$6a_3 - 6a_2 + 2a_1 = 0 \tag{40}$$

and so $a_3 = a_2 = -a_0 = -1$. Therefore the first three nonzero terms are

$$y = 1 - x^2 - x^3 + \dots. \tag{41}$$