2E2 Tutorial Sheet 16 Second Term, Solutions[1]

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1. (3) Assuming the solution of

$$(1-x)y' + y = 0 \tag{1}$$

has a series expansion about $x = 0$ work out the recursion relation. Write out the first few terms and show that the series terminates to give $y = A(1-x)$ for arbitrary A.

Solution: So we begin by writing

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

and so by differentiation we get

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} \tag{3}$$

and hence

$$xy' = \sum_{n=0}^{\infty} a_n n x^n. \tag{4}$$

Thus, substituting the differential equation we get

$$\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \tag{5}$$

In order to make progress we need to rewrite the first of these three series so that it is in the form

$$\sum_{n=0}^{\infty} \text{stuff}_n x^n \tag{6}$$

so that all three bits in the equation match. Well, let $m = n - 1$ in the expression for y', (3), to get

$$y' = \sum_{m=0}^{\infty} a_{m+1}(m+1)x^m. \tag{7}$$

In fact, this looks at first like it gives

$$y' = \sum_{m=-1}^{\infty} a_{m+1}(m+1)x^m \tag{8}$$

[1]Conor Houghton, `houghton@maths.tcd.ie` and `http://www.maths.tcd.ie/~houghton/ 2E2.html`

but the $m = -1$ term is zero, so that's fine. Now m is just an index so we can rename it n, don't get confused, this isn't the original n, we just want all parts of the equation to look the same.

In fact, we now have

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \tag{9}$$

and we can group this all together to give

$$\sum_{n=0}^{\infty} [a_{n+1}(n+1) + (1-n)a_n]x^n = 0. \tag{10}$$

The recursion relation is

$$a_{n+1} = -\left(\frac{1-n}{1+n}\right) a_n \tag{11}$$

and this applies to n from zero upwards since that is what appears in the sum sign.

Starting at $n = 0$ we have

$$a_1 = -a_0. \tag{12}$$

For $n = 1$ we get

$$a_2 = 0 \tag{13}$$

and the series terminates here because every term is something multiplied by the one before and so if a_2 is zero the rest of the series is zero. Thus $y = a_0(1-x)$ for arbitrary a_0.

2. (3) Assuming the solution of

$$(1-x^2)y' - 2xy = 0 \tag{14}$$

has a series expansion about $x = 0$, work out the recursion relation and write out the first four non-zero terms.

Solution: Assuming the solution of

$$(1-x^2)y' - 2xy = 0 \tag{15}$$

has a series expansion about $x = 0$, work out the recursion relation and write out the first four non-zero terms.

Answer: Once again let

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{16}$$

and, as before,

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} \tag{17}$$

so

$$x^2 y' = \sum_{n=0}^{\infty} a_n n x^{n+1} \tag{18}$$

and finally

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1}. \tag{19}$$

The equation then reads

$$\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1}. \tag{20}$$

Once again, the first term is a problem because it doesn't have the same form as the other two. So, take

$$\sum_{n=0}^{\infty} a_n n x^{n-1} \tag{21}$$

and put $n-1 = m+1$ and, hence, $n = m+2$. When $n = 0$, $m = -2$ and when $n = 1$, $m = -1$. Thus

$$\sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{m=-2}^{\infty} a_{m+2}(m+2)x^{m+1} \tag{22}$$

and, once again renaming m as n we get

$$\sum_{n=-2}^{\infty} (n+2)a_{n+2}x^{n+1} - \sum_{n=0}^{\infty} n a_n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0. \tag{23}$$

The problem now is with the range that the first sum runs over. The $n = -2$ term is no problem, it is zero, but the $n = -1$ term is a_1. Thus, we write

$$\sum_{n=-2}^{\infty} (n+2)a_{n+2}x^{n+1} = a_1 + \sum_{n=0}^{\infty}(n+2)a_{n+2}x^{n+1} \tag{24}$$

and the equation becomes

$$a_1 + \sum_{n=0}^{\infty}(n+2)a_{n+2}x^{n+1} - \sum_{n=0}^{\infty} a_n n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0. \tag{25}$$

Thus

$$a_1 + \sum_{n=0}^{\infty}[(n+2)a_{n+2} - na_n - 2a_n]x^{n+1} = 0. \tag{26}$$

Notice that the summand starts with the x term. The recursion relation is therefore

$$a_{n+2} = a_n \tag{27}$$

with the additional conditions $a_1 = 0$. Hence, $a_6 = a_4 = a_2 = a_0$, $a_5 = a_3 = a_1 = 0$ and so on. The first four nonzero terms of the expansion gives

$$y = a_0(1 + x^2 + x^4 + x^6 + \ldots). \tag{28}$$

3. (2) Assuming the solution of

$$y'' - 3y' + 2y = 0 \tag{29}$$

has a series expansion about $x = 0$, by substitution, work out the recursion relation. If $y(0) = 1$ and $y'(0) = 0$ what are the first three non-zero terms.

Solution: Again

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{30}$$

so

$$y' = \sum_{n=0}^{\infty} na_n x^{n-1} \tag{31}$$

and

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} \tag{32}$$

Thus,

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 3\sum_{n=0}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0 \tag{33}$$

Again, we want to make each part look the same. As before, changing the index gi ves

$$y' = \sum_{n=0}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n. \tag{34}$$

The same thing can be done with the y'': let $m = n - 2$ to get

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=-2}^{\infty}(m+1)(m+2)a_{m+2}x^m \tag{35}$$

and the $m = -2$ and $m = -1$ terms are both zero, so, renaming the m as n we get

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - 3\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0 \tag{36}$$

and this gives

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n]x^n = 0. \tag{37}$$

The recursion relation is

$$(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0. \tag{38}$$

Now apply the initial conditions, $y(0) = 1$ implies that $a_0 = 1$, $y'(0) = 0$ implies $a_1 = 0$. For $n = 0$ the recursion relation gives

$$2a_2 - 3a_1 + 2a_0 = 0 \tag{39}$$

and so $a_2 = -a_0 = -1$. Next $n = 1$ gives

$$6a_3 - 6a_2 + 2a_1 = 0 \tag{40}$$

and so $a_3 = a_2 = -a_0 = -1$. Therefore the first three nonzero terms are

$$y = 1 - x^2 - x^3 + \ldots. \tag{41}$$