2E2 Tutorial Sheet 15 Second Term, Solutions[1]

17 February 2004

1. (3) By linearizing around the critical points, draw the phase plane portrait of

$$y'' + y - y^3 = 0 \tag{1}$$

Solution: First, rewrite in first order form so let $y_1 = y$ and define $y_2 = y_1'$, now, from the equation $y_1'' = y_2' = y_1^3 - y_1$, putting these together gives:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1^3 - y_1 \end{aligned} \tag{2}$$

The stationary points occur when $y_1' = y_2' = 0$, hence $y_2 = 0$ and $y_1^3 - y_1 = 0$, or, when $y_1 = -1$ 0r $y_1 = 0$ or $y_1 = 1$. We will look at each of these stationary points in turn.

Near $y_1 = -1$ and $y_2 = 0$ we have $y_1 = -1 + \eta$ where η is small. Hence

$$y_2' = (-1+\eta)^3 - (-1+\eta) \approx -1 + 3\eta + 1 - \eta = 2\eta \tag{3}$$

and so, near this stationary point, the system is approximately

$$\begin{aligned} \eta' &= y_2 \\ y_2' &\approx 2\eta \end{aligned} \tag{4}$$

or

$$\begin{pmatrix} \eta \\ y_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ y_2 \end{pmatrix} \tag{5}$$

The matrix here has eigenvalues $\lambda_1 = \sqrt{2}$ and $\lambda_2 = -\sqrt{2}$ with eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \tag{6}$$

and

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \tag{7}$$

Hence, this stationary point is a saddle point and provided η remains small, it is approximated by

$$\begin{pmatrix} \eta \\ y_2 \end{pmatrix} \approx C_1 \mathbf{x}_1 e^{\sqrt{2}t} + C_2 \mathbf{x}_2 e^{-\sqrt{2}t} \tag{8}$$

[1] Conor Houghton, `houghton@maths.tcd.ie` and `http://www.maths.tcd.ie/~houghton/ 2E2.html`

Near $y_1 = 0$ and $y_2 = 0$ we assume both y_1 and y_2 are small and make the approx mation

$$y_2' = y_1^3 - y_1 \approx -y_1 \tag{9}$$

and so, near this stationary point, the system is approximately

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{10}$$

This matrix has eigenvalues $\pm i$ and so this is a circle node.

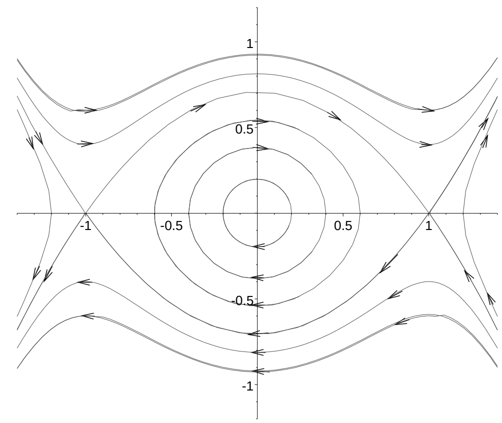
Near $y_1 = 1$ and $y_2 = 0$ we have $y_1 = 1 + \eta$ where η is small. Hence

$$y_2' = (1+\eta)^3 - (1+\eta) \approx 2\eta \tag{11}$$

and so, near this stationary point, the system is approximately

$$\begin{pmatrix} \eta \\ y_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ y_2 \end{pmatrix} \tag{12}$$

and so this point is the same as the $y_1 = -1$, $y_2 = 0$ stationary point.



2. (3) By linearizing around the critical points, draw the phase plane portrait of

$$y'' = \cos 2y \tag{13}$$

Solution: As before, rewrite as a first order system:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= \cos 2y_1 \end{aligned} \tag{14}$$

now, the critical points are located where $y_1' = y_2' = 0$. This happens when $y_2 = 0$ and $\cos 2y_1 = 0$, that means $2y_1 = n\pi/2$ where n is an odd integer, or $y_1 = n\pi/4$ where again n is an odd integer.

Near $y_1 = \pi/4$ write $y_1 = \pi/4 + \eta$ and use $\cos 2y_1 = \cos 2(\pi/4 + eta) = -\sin 2\eta$ and this linearizes as $\sin 2\eta \sim 2\eta$ so the system become s

$$\begin{aligned} \eta' &= y_2 \\ y_2' &= -2\eta. \end{aligned} \tag{15}$$

This is a center. The matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \tag{16}$$

and so, by calculating the eigenvalues and eigenvectors, the general solution is

$$\begin{pmatrix} \eta \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ sqrt2i \end{pmatrix} e^{\sqrt{2}it} + c_2 \begin{pmatrix} 1 \\ -\sqrt{2}i \end{pmatrix} e^{-\sqrt{2}it} \tag{17}$$

and by beginning at $\eta = r$ and $y_2 = 0$ we get

$$\begin{pmatrix} \eta \\ y_2 \end{pmatrix} = r \begin{pmatrix} \cos\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t \end{pmatrix} \tag{18}$$

so the saddle point is an ellipse with the vertical $\sqrt{2}$ times as long as the horizontal.

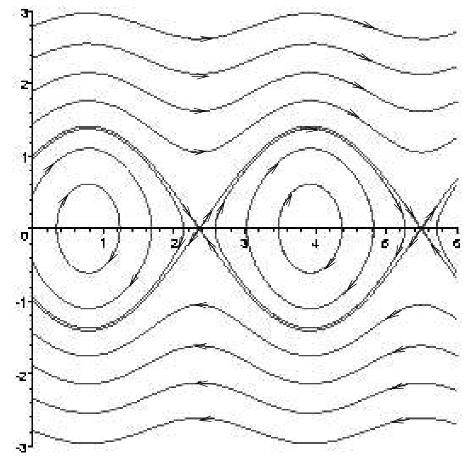
Near $y_1 = 3\pi/4$ write $y_1 = 3\pi/4 + \eta$ and use $\cos 2y_1 = \cos 2(3\pi/4 + eta) = \sin 2\eta$ and this linearizes as $\sin 2\eta \sim 2\eta$ so the system beco mes

$$\begin{aligned} \eta' &= y_2 \\ y_2' &= 2\eta \end{aligned} \tag{19}$$

This is a saddle-point with eigenvalues $\pm\sqrt{2}$ and eigenvectors

$$\begin{pmatrix} 1 \\ \pm\sqrt{2} \end{pmatrix}. \tag{20}$$

This pattern repeats by periodicity, the phase portrait is



3. (3) By linearizing, sketch the phase space trajectories of

$$y'' = -y' + y - y^2 \quad (21$$

Solution: The first order system in this case is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_2 + y_1(1 - y_1). \end{aligned} \quad (22$$

This has two same critical points, one at $(y_1, y_2) = (0, 0)$ and the second at (y_1, y_2) $(1, 0)$.

Near $(0, 0)$ the system linearizes to the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 - y_2. \end{aligned} \quad (23$$

It is a saddlepoint. It is slightly different to the saddlepoint that was here i n questio one, in this case $\lambda_1 = -(1 + \sqrt{5})/2$ with eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} -2 \\ 1 + \sqrt{5} \end{pmatrix} \quad (24$$

and $\lambda_2 = (-1 + \sqrt{5})/2$ with eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} 2 \\ -1 + \sqrt{5} \end{pmatrix} \quad (25$$

Near $(1,0)$ write $y_1 = 1 + \eta$ to get

$$\begin{aligned} \eta' &= y_2 \\ y_2' &= -\eta - y_2 \end{aligned} \tag{26}$$

so the eigenvalues are

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \tag{27}$$

so this is an inward moving exponential spiral.

To draw the phase plane, draw the saddlepoint and the circle and try to join the m up. The answer is