1. (3) By linearizing around the critical points, draw the phase plane portrait of

\[ y'' + y - y^3 = 0 \]  

Solution: First, rewrite in first order form so let \( y_1 = y \) and define \( y_2 = y'_1 \), now, from the equation \( y'_1 = y'_2 = y_1^3 - y_1 \), putting these together gives:

\[
\begin{align*}
y'_1 &= y_2 \\
y'_2 &= y_1^3 - y_1
\end{align*}
\]

The stationary points occur when \( y'_1 = y'_2 = 0 \), hence \( y_2 = 0 \) and \( y_1^3 - y_1 = 0 \), or, when \( y_1 = -1 \) or \( y_1 = 0 \) or \( y_1 = 1 \). We will look at each of these stationary points in turn.

Near \( y_1 = 1 \) and \( y_2 = 0 \) we have \( y_1 = 1 + \eta \) where \( \eta \) is small. Hence

\[
y'_2 = (-1 + \eta)^3 - (-1 + \eta) \approx -1 + 3\eta + 1 - \eta = 2\eta
\]

and so, near this stationary point, the system is approximately

\[
\begin{align*}
y'_1 &= y_2 \\
y'_2 &\approx 2\eta
\end{align*}
\]

or

\[
\begin{pmatrix} \eta \\ y_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ y_2 \end{pmatrix}
\]

The matrix here has eigenvalues \( \lambda_1 = \sqrt{2} \) and \( \lambda_2 = -\sqrt{2} \) with eigenvectors

\[
x_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}
\]

and

\[
x_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}
\]

Hence, this stationary point is a saddle point and provided \( \eta \) remains small, it is approximated by

\[
\begin{pmatrix} \eta \\ y_2 \end{pmatrix} \approx C_1x_1e^{\sqrt{2}t} + C_2x_2e^{-\sqrt{2}t}
\]

Near \( y_1 = 0 \) and \( y_2 = 0 \) we assume both \( y_1 \) and \( y_2 \) are small and make the approximation

\[
y'_2 = y_1^3 - y_1 \approx -y_1
\]

and so, near this stationary point, the system is approximately

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\]

This matrix has eigenvalues \( \pm i \) and so this is a circle node.

Near \( y_1 = 1 \) and \( y_2 = 0 \) we have \( y_1 = 1 + \eta \) where \( \eta \) is small. Hence

\[
y'_2 = (1 + \eta)^3 - (1 + \eta) \approx 2\eta
\]

and so, near this stationary point, the system is approximately

\[
\begin{pmatrix} \eta \\ y_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ y_2 \end{pmatrix}
\]

and so this point is the same as the \( y_1 = 1, y_2 = 0 \) stationary point.

2. (3) By linearizing around the critical points, draw the phase plane portrait of

\[ y'' = \cos 2y \]

Solution: As before, rewrite as a first order system:

\[
\begin{align*}
y'_1 &= y_2 \\
y'_2 &= \cos 2y_1
\end{align*}
\]
now, the critical points are located where $y'_1 = y'_2 = 0$. This happens when $y_2 = 0$ and $\cos 2y_1 = 0$, that means $2y_1 = n\pi/2$ where $n$ is an odd integer, or $y_1 = n\pi/4$ where again $n$ is an odd integer.

Near $y_1 = \pi/4$ write $y_1 = \pi/4 + \eta$ and use $\cos 2y_1 = \cos 2(\pi/4 + \epsilon a) = -\sin 2\eta$ and this linearizes as $\sin 2\eta \sim 2\eta$ so the system become

$$
\begin{align*}
\eta' &= y_2 \\
y'_2 &= -2\eta.
\end{align*}
$$

(15)

This is a center. The matrix is

$$
A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}
$$

(16)

and so, by calculating the eigenvalues and eigenvectors, the general solution is

$$
\begin{pmatrix} \eta \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \text{sqr}t2i \end{pmatrix} e^{\sqrt{2}it} + c_2 \begin{pmatrix} 1 \\ -\sqrt{2}i \end{pmatrix} e^{-\sqrt{2}it}
$$

(17)

and by beginning at $\eta = r$ and $y_2 = 0$ we get

$$
\begin{pmatrix} \eta \\ y_2 \end{pmatrix} = r \begin{pmatrix} \cos \sqrt{2}t \\ -\sqrt{2}\sin \sqrt{2}t \end{pmatrix}
$$

(18)

so the saddle point is an ellipse with the vertical $\sqrt{2}$ times as long as the horizontal.

Near $y_1 = 3\pi/4$ write $y_1 = 3\pi/4 + \eta$ and use $\cos 2y_1 = \cos 2(3\pi/4 + \epsilon a) = \sin 2\eta$ and this linearizes as $\sin 2\eta \sim 2\eta$ so the system becomes

$$
\begin{align*}
\eta' &= y_2 \\
y'_2 &= y_1 \\
y''_1 &= y_2 + y_1.
\end{align*}
$$

(19)

This is a saddle-point with eigenvalues $\pm \sqrt{2}$ and eigenvectors

$$
\begin{pmatrix} 1 \\ \pm \sqrt{2} \end{pmatrix}.
$$

(20)

This pattern repeats by periodicity, the phase portrait is

3. (3) By linearizing, sketch the phase space trajectories of

$$
y'' = -y' + y - y^2
$$

(21)

Solution: The first order system in this case is

$$
\begin{align*}
y'_1 &= y_2 \\
y'_2 &= -y_2 + y_1(1 - y_1).
\end{align*}
$$

(22)

This has two same critical points, one at $(y_1, y_2) = (0, 0)$ and the second at $(y_1, y_2) = (1, 0)$.

Near $(0, 0)$ the system linearizes to the system

$$
\begin{align*}
y'_1 &= y_2 \\
y'_2 &= y_1 - y_2.
\end{align*}
$$

(23)

It is a saddlepoint. It is slightly different to the saddlepoint that was here in question one, in this case $\lambda_1 = -(1 + \sqrt{5})/2$ with eigenvector

$$
\begin{pmatrix} -2 \\ 1 + \sqrt{5} \end{pmatrix}
$$

(24)

and $\lambda_2 = (-1 + \sqrt{5})/2$ with eigenvector

$$
\begin{pmatrix} -2 \\ -1 + \sqrt{5} \end{pmatrix}
$$

(25)
Near $(1, 0)$ write $y_1 = 1 + \eta$ to get
\[
\begin{align*}
\eta' &= y_2 \\
y_2' &= -\eta - y_2
\end{align*}
\] (26)
so the eigenvalues are
\[
\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i
\] (27)
so this is an inward moving exponential spiral.
To draw the phase plane, draw the saddlepoint and the circle and try to join the m up. The answer is