1. (3) By linearizing around the critical points, draw the phase plane portrait of

\[ y'' + y - y^3 = 0 \]  \hspace{1cm} (1)

*Solution:* First, rewrite in first order form so let \( y_1 = y \) and define \( y_2 = y'_1 \), now, from the equation \( y''_1 = y'_2 = y^3_1 - y_1 \), putting these together gives:

\[
\begin{align*}
    y'_1 &= y_2 \\
    y'_2 &= y^3_1 - y_1
\end{align*}
\]  \hspace{1cm} (2)

The stationary points occur when \( y'_1 = y'_2 = 0 \), hence \( y_2 = 0 \) and \( y^3_1 - y_1 = 0 \), or, when \( y_1 = -1 \) or \( y_1 = 0 \) or \( y_1 = 1 \). We will look at each of these stationary points in turn.

Near \( y_1 = -1 \) and \( y_2 = 0 \) we have \( y_1 = -1 + \eta \) where \( \eta \) is small. Hence

\[ y'_2 = (-1 + \eta)^3 - (-1 + \eta) \approx -1 + 3\eta + 1 - \eta = 2\eta \]  \hspace{1cm} (3)

and so, near this stationary point, the system is approximately

\[
\begin{align*}
    \eta' &= y_2 \\
    y'_2 &\approx 2\eta
\end{align*}
\]  \hspace{1cm} (4)

or

\[
\begin{pmatrix} \eta \\ y_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ y_2 \end{pmatrix} \]  \hspace{1cm} (5)

The matrix here has eigenvalues \( \lambda_1 = \sqrt{2} \) and \( \lambda_2 = -\sqrt{2} \) with eigenvectors

\[
x_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \]  \hspace{1cm} (6)

and

\[
x_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \]  \hspace{1cm} (7)

Hence, this stationary point is a saddle point and provided \( \eta \) remains small, it is approximated by

\[
\begin{pmatrix} \eta \\ y_2 \end{pmatrix} \approx C_1 x_1 e^{\sqrt{2}t} + C_2 x_2 e^{-\sqrt{2}t} \]  \hspace{1cm} (8)
Near $y_1 = 0$ and $y_2 = 0$ we assume both $y_1$ and $y_2$ are small and make the approximation
\[ y'_2 = y_1^3 - y_1 \approx -y_1 \] (9)
and so, near this stationary point, the system is approximately
\[ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \] (10)
This matrix has eigenvalues $\pm i$ and so this is a circle node.

Near $y_1 = 1$ and $y_2 = 0$ we have $y_1 = 1 + \eta$ where $\eta$ is small. Hence
\[ y'_2 = (1 + \eta)^3 - (1 + \eta) \approx 2\eta \] (11)
and so, near this stationary point, the system is approximately
\[ \begin{pmatrix} \eta \\ y_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ y_2 \end{pmatrix} \] (12)
and so this point is the same as the $y_1 = -1$, $y_2 = 0$ stationary point.

2. (3) By linearizing around the critical points, draw the phase plane portrait of
\[ y'' = \cos 2y \] (13)

Solution: As before, rewrite as a first order system:
\[ \begin{align*}
    y'_1 &= y_2 \\
    y'_2 &= \cos 2y_1
\end{align*} \] (14)
now, the critical points are located where \( y'_1 = y'_2 = 0 \). This happens when \( y_2 = 0 \) and \( \cos 2y_1 = 0 \), that means \( 2y_1 = n\pi/2 \) where \( n \) is an odd integer, or \( y_1 = n\pi/4 \) where again \( n \) is an odd integer.

Near \( y_1 = \pi/4 \) write \( y_1 = \pi/4 + \eta \) and use \( \cos 2y_1 = \cos 2(\pi/4 + \eta) = -\sin 2\eta \) and this linearizes as \( \sin 2\eta \sim 2\eta \) so the system become:

\[
\begin{align*}
\eta' &= y_2 \\
y'_2 &= -2\eta.
\end{align*}
\]  

This is a center. The matrix is

\[
A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}
\]  

and so, by calculating the eigenvalues and eigenvectors, the general solution is

\[
\begin{pmatrix} \eta \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \sqrt{2}i \end{pmatrix} e^{\sqrt{2}it} + c_2 \begin{pmatrix} 1 \\ -\sqrt{2}i \end{pmatrix} e^{-\sqrt{2}it}
\]  

and by beginning at \( \eta = r \) and \( y_2 = 0 \) we get

\[
\begin{pmatrix} \eta \\ y_2 \end{pmatrix} = r \begin{pmatrix} \cos \sqrt{2}t \\ -\sqrt{2}\sin \sqrt{2}t \end{pmatrix}
\]  

so the saddle point is an ellipse with the vertical \( \sqrt{2} \) times as long as the horizontal.

Near \( y_1 = 3\pi/4 \) write \( y_1 = 3\pi/4 + \eta \) and use \( \cos 2y_1 = \cos 2(3\pi/4 + \eta) = \sin 2\eta \) and this linearizes as \( \sin 2\eta \sim 2\eta \) so the system becomes

\[
\begin{align*}
\eta' &= y_2 \\
y'_2 &= 2\eta
\end{align*}
\]  

This is a saddle-point with eigenvalues \( \pm \sqrt{2} \) and eigenvectors

\[
\begin{pmatrix} 1 \\ \pm \sqrt{2} \end{pmatrix}
\]  

This pattern repeats by periodicity, the phase portrait is
3. (3) By linearizing, sketch the phase space trajectories of

\[ y'' = -y' + y - y^2 \]  \hspace{1cm} (21)

**Solution:** The first order system in this case is

\[
\begin{align*}
y_1' &= y_2 \\
y_2' &= -y_2 + y_1(1 - y_1). \hspace{1cm} (22)
\end{align*}
\]

This has two same critical points, one at \((y_1, y_2) = (0, 0)\) and the second at \((y_1, y_2) = (1, 0)\).

Near \((0, 0)\) the system linearizes to the system

\[
\begin{align*}
y_1' &= y_2 \\
y_2' &= y_1 - y_2. \hspace{1cm} (23)
\end{align*}
\]

It is a saddlepoint. It is slightly different to the saddlepoint that was here in question one, in this case \(\lambda_1 = -(1 + \sqrt{5})/2\) with eigenvector

\[ x_1 = \begin{pmatrix} -2 \\ 1 + \sqrt{5} \end{pmatrix} \] \hspace{1cm} (24)

and \(\lambda_2 = (-1 + \sqrt{5})/2\) with eigenvector

\[ x_2 = \begin{pmatrix} 2 \\ -1 + \sqrt{5} \end{pmatrix} \] \hspace{1cm} (25)
Near \((1, 0)\) write \(y_1 = 1 + \eta\) to get

\[
\begin{align*}
\eta' &= y_2 \\
y_2' &= -\eta - y_2
\end{align*}
\]  

(26)

so the eigenvalues are

\[
\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i
\]

(27)

so this is an inward moving exponential spiral.

To draw the phase plane, draw the saddlepoint and the circle and try to join the m up. The answer is