2E2 Tutorial Sheet 14 Second Term, Solutions¹

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Consider the non-linear differential equation

$$y'' = y - y^2 \tag{1}$$

- 1. (1) By defining $y_1 = y$ and $y_2 = y'_1$ convert this into two first order equations.
- 2. (1) The stationary points are the points where $y'_1 = y'_2 = 0$, find the two stationary points for this equation.
- 3. (2) Consider the $y_1 = 0$ stationary point, linearize the equations near this point by assuming $y_1 \ll 1$. Solve the corresponding linear equations. What sort of stationary point is this?
- 4. (2) Consider the $y_1 = 1$ stationary point, linearize the equations near this point by assuming $y_1 = 1 + \eta$ where $\eta \ll 1$. Solve the corresponding linear equations. What sort of stationary point is this?
- 5. (2) Try and draw the whole phase diagram, first draw in the two stationary points and then try and join the lines, remember the lines don't cross.

Solution: First we change the system into a pair of first order equations, $y_1 = y$ and

$$y'_1 = y_2$$

 $y'_2 = y_1(1 - y_1).$ (2)

Setting $y_1' = y_2' = 0$ gives $y_2 = 0$ and $y_1(y_1 - 1) = 0$ so this has two critical points, one at $(y_1, y_2) = (0, 0)$ and the second at $(y_1, y_2) = (1, 0)$.

Near (0,0) the system linearizes to the system

$$y'_1 = y_2$$

 $y'_2 = y_1$. (3)

which has eigenvalue $\lambda_1 = 1$ corresponding to eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} 1\\1 \end{pmatrix} \tag{4}$$

and eigenvalue $\lambda_1 = -1$ corresponding to eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5}$$

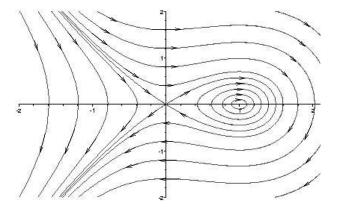
It is a saddlepoint.

Near (1,0) write $y_1 = 1 + \eta$ to get

$$\begin{aligned}
 \eta' &= y_2 \\
 y_2' &= -\eta
 \end{aligned}$$

so the eigenvalues are $\lambda = \pm i$ and the critical point is a center.

To draw the phase plane, draw the saddlepoint and the circle and try to join the m up. The answer is



¹Conor Houghton, houghton@maths.tcd.ie and http://www.maths.tcd.ie/~houghton/ 2E2.html