2E2 Tutorial Sheet 14 Second Term, Solutions[1]

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Consider the non-linear differential equation

$$y'' = y - y^2 \tag{1}$$

1. (1) By defining $y_1 = y$ and $y_2 = y_1'$ convert this into two first order equations.
2. (1) The stationary points are the points where $y_1' = y_2' = 0$, find the two stationary points for this equation.
3. (2) Consider the $y_1 = 0$ stationary point, linearize the equations near this point by assuming $y_1 \ll 1$. Solve the corresponding linear equations. What sort of stationary point is this?
4. (2) Consider the $y_1 = 1$ stationary point, linearize the equations near this point by assuming $y_1 = 1 + \eta$ where $\eta \ll 1$. Solve the corresponding linear equations. What sort of stationary point is this?
5. (2) Try and draw the whole phase diagram, first draw in the two stationary points and then try and join the lines, remember the lines don't cross.

Solution: First we change the system into a pair of first order equations, $y_1 = y$ and

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1(1 - y_1). \end{aligned} \tag{2}$$

Setting $y_1' = y_2' = 0$ gives $y_2 = 0$ and $y_1(y_1 - 1) = 0$ so this has two critical points, one at $(y_1, y_2) = (0, 0)$ and the second at $(y_1, y_2) = (1, 0)$.

Near $(0, 0)$ the system linearizes to the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1. \end{aligned} \tag{3}$$

which has eigenvalue $\lambda_1 = 1$ corresponding to eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4}$$

and eigenvalue $\lambda_1 = -1$ corresponding to eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5}$$

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It is a saddlepoint.

Near $(1,0)$ write $y_1 = 1 + \eta$ to get

$$\begin{aligned} \eta' &= y_2 \\ y_2' &= -\eta \end{aligned} \tag{6}$$

so the eigenvalues are $\lambda = \pm i$ and the critical point is a center.

To draw the phase plane, draw the saddlepoint and the circle and try to join the m up. The answer is