2E2 Tutorial Sheet 12 Second Term[1]

27 January 2004

1. (2) Last week the last question was to find the solution of

$$\frac{dy_1}{dt} = -y_1 - 2y_2 \tag{1}$$

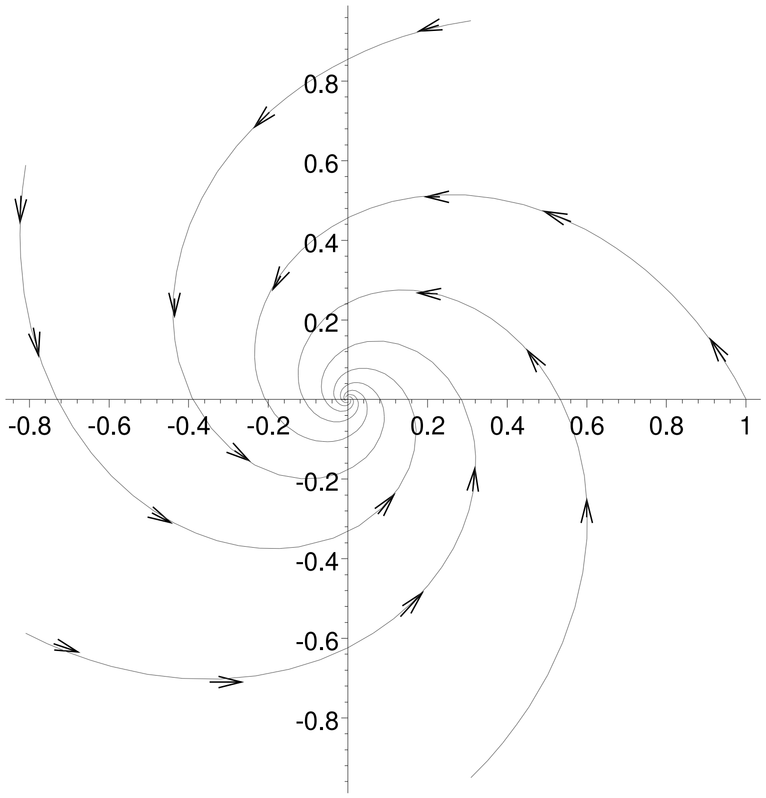
$$\frac{dy_2}{dt} = 2y_1 - y_2 \tag{2}$$

and put it in real form. The answer was

$$\mathbf{y} = \begin{pmatrix} r\cos 2t \\ r\sin 2t \end{pmatrix} e^{-t} \tag{3}$$

Plot the phase plane diagram for this.

Solution: So, this gives the inward spiral. Notice how fast the spiral goes in. The radius decreases exponentially.



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2. (3) Find the general solution for the system

$$\begin{aligned} \frac{dy_1}{dt} &= 3y_1 + y_2 && (4) \\ \frac{dy_2}{dt} &= -y_1 + y_2 && (5) \end{aligned}$$

Solution: This is one of those systems where there is only one eigenvalue and only one eigenvector, $\lambda = 2$ with

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6}$$

so the solution is of the form

$$\mathbf{y} = c_1 \mathbf{x} e^{2t} + c_2 \left(t\mathbf{x} + \mathbf{u} \right) e^{2t} \tag{7}$$

where you need to find $\mathbf{u}$ by substituting

$$\mathbf{y} = \left(t\mathbf{x} + \mathbf{u} \right) e^{2t} \tag{8}$$

back into the equation. This means the $\mathbf{u}$ vector in the extra solution is the solution to

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Writing

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$$

gives equations

$$\begin{aligned} a + b &= 1 \\ -a - b &= -1 \end{aligned}$$

These two equations are the same, as you expect, and if $b = 0$ then $a = 1$. Thus, the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{2t}$$

or

$$\mathbf{y} = \left[(c_1 + c_2 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{2t}.$$

3. (3) Find the solution for the system

$$\begin{aligned} y_1' &= 4y_1 + y_2 \\ y_2' &= -y_1 + 2y_2. \end{aligned}$$

with initial conditions $y_1(0) = 3$ and $y_2(0) = 2$.

Solution:

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \tag{9}$$

and there is only one eigenvector,

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{10}$$

with eigenvalue $\lambda = 3$. The solution is

$$\mathbf{y} = c_1 \mathbf{x} e^{\lambda t} + c_2(t\mathbf{x} + \mathbf{u})e^{\lambda t} \tag{11}$$

where $\mathbf{u}$ satisfies

$$(A - \lambda \mathbf{1})\,\mathbf{u} = \mathbf{x} \tag{12}$$

and so, in this case,

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{13}$$

and a solution to this is

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{14}$$

and so the solution is

$$\mathbf{y} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \left[t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{3t} \tag{15}$$

Now, putting $t = 0$ we get

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{16}$$

and, hence,

$$\begin{aligned} 3 &= -c_1 - c_2 \\ 2 &= c_1 \end{aligned} \tag{17}$$

aso $c_2 = 1$ and $c_2 = -5$ giving

$$\mathbf{y} = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} - 5 \left[t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{3t} \tag{18}$$

or

$$\begin{aligned} y_1 &= (3 + 5t)e^{3t} \\ y_2 &= (2 - 5t)e^{3t} \end{aligned} \tag{19}$$