27 January 2004

1. (2) Last week the last question was to find the solution of

\[ \frac{dy_1}{dt} = -y_1 - 2y_2 \]  
\[ \frac{dy_2}{dt} = 2y_1 - y_2 \]  

and put it in real form. The answer was

\[ y = \begin{pmatrix} r \cos 2t \\ r \sin 2t \end{pmatrix} e^{-t} \]

Plot the phase plane diagram for this.

Solution: So, this gives the inward spiral. Notice how fast the spiral goes in. The radius decreases exponentially.

\[ \begin{array}{c}
\cdot \\
\cdot
\end{array} \]

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\[ ^1 \text{Conor Houghton, houghton@maths.tcd.ie and http://www.maths.tcd.ie/~houghton/ 2E2.html} \]
2. (3) Find the general solution for the system
\[
\begin{align*}
\frac{dy_1}{dt} &= 3y_1 + y_2 \\
\frac{dy_2}{dt} &= -y_1 + y_2
\end{align*}
\]

**Solution:** This is one of those systems where there is only one eigenvalue and only one eigenvector, \( \lambda = 2 \) with
\[
\begin{pmatrix}
1 \\
-1
\end{pmatrix}
\]
so the solution is of the form
\[
y = c_1 x e^{2t} + c_2 (t x + u) e^{2t}
\]
where you need to find \( u \) by substituting
\[
y = (t x + u) e^{2t}
\]
back into the equation. This means the \( u \) vector in the extra solution is the solution to
\[
\begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
= \begin{pmatrix}
1 \\
-1
\end{pmatrix}.
\]
Writing
\[
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix} = \begin{pmatrix}
a \\
b
\end{pmatrix}
\]
gives equations
\[
\begin{align*}
a + b &= 1 \\
-a - b &= -1
\end{align*}
\]
These two equations are the same, as you expect, and if \( b = 0 \) then \( a = 1 \). Thus, the general solution is
\[
y = c_1 \begin{pmatrix}
1 \\
-1
\end{pmatrix} e^{2t} + c_2 \left[ \begin{pmatrix}
1 \\
-1
\end{pmatrix} t + \begin{pmatrix}
1 \\
0
\end{pmatrix} \right] e^{2t}
\]
or
\[
y = \left[ (c_1 + c_2 t) \begin{pmatrix}
1 \\
-1
\end{pmatrix} + c_2 \begin{pmatrix}
1 \\
0
\end{pmatrix} \right] e^{2t}.
\]

3. (3) Find the solution for the system
\[
\begin{align*}
y_1' &= 4y_1 + y_2 \\
y_2' &= -y_1 + 2y_2.
\end{align*}
\]
with initial conditions \( y_1(0) = 3 \) and \( y_2(0) = 2 \).

*Solution:*

\[
A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}
\]  
(9)

and there is only one eigenvector,

\[
x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]  
(10)

with eigenvalue \( \lambda = 3 \). The solution is

\[
y = c_1 x e^{\lambda t} + c_2 (tx + u) e^{\lambda t}
\]
(11)

where \( u \) satisfies

\[
(A - \lambda I) u = x
\]
(12)

and so, in this case,

\[
\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} u = \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]
(13)

and a solution to this is

\[
u = \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\]
(14)

and so the solution is

\[
y = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \left[ t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{3t}
\]
(15)

Now, putting \( t = 0 \) we get

\[
\begin{pmatrix} 3 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\]
(16)

and, hence,

\[
\begin{align*}
3 &= -c_1 - c_2 \\
2 &= c_1
\end{align*}
\]
(17)

aso \( c_2 = 1 \) and \( c_2 = -5 \) giving

\[
y = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} - 5 \left[ t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{3t}
\]
(18)

or

\[
\begin{align*}
y_1 &= (3 + 5t)e^{3t} \\
y_2 &= (2 - 5t)e^{3t}
\end{align*}
\]
(19)