2E2 Tutorial Sheet 11 Second Term, Solutions[1]

18 January 2003

1. (2) Last week you were asked to find the solution for the system

$$\frac{dy_1}{dt} = -3y_1 + 2y_2$$
$$\frac{dy_2}{dt} = -2y_1 + 2y_2$$

The solution is

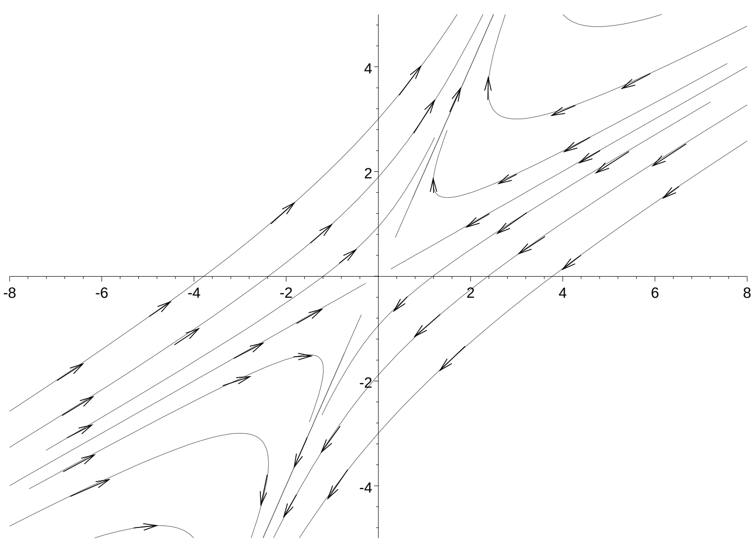
$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} \tag{1}$$

Now draw the phase diagram for this solution and name the type of stationary point (saddlepoint or outward improper.)

Solution: So, any point that starts on the

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2}$$

eigenvector will move inwards, since $c_1 = 0$ and $c^2 \exp -2t$ gets small as t increases, anywhere on the other eigenvectors will move straight outwards. If you aren't on either eigenvector, the amount along the negative eigenvalue eigenvector decreases and the amount along the positive eigenvector eigenvalue increases and so you move outwards getting closer and closer to the positive eigenvalue line. The phase diagram is



[1]Conor Houghton, `houghton@maths.tcd.ie` and `http://www.maths.tcd.ie/~houghton/ 2E2.html`

where the arrows go outwards except on the line defined by $\mathbf{x}_2$. The stationary point is a saddle point.

2. (2) Last week you were asked to find the solution for the system

$$\frac{dy_1}{dt} = 3y_1 + y_2 \tag{3}$$
$$\frac{dy_2}{dt} = y_1 + 3y_2 \tag{4}$$

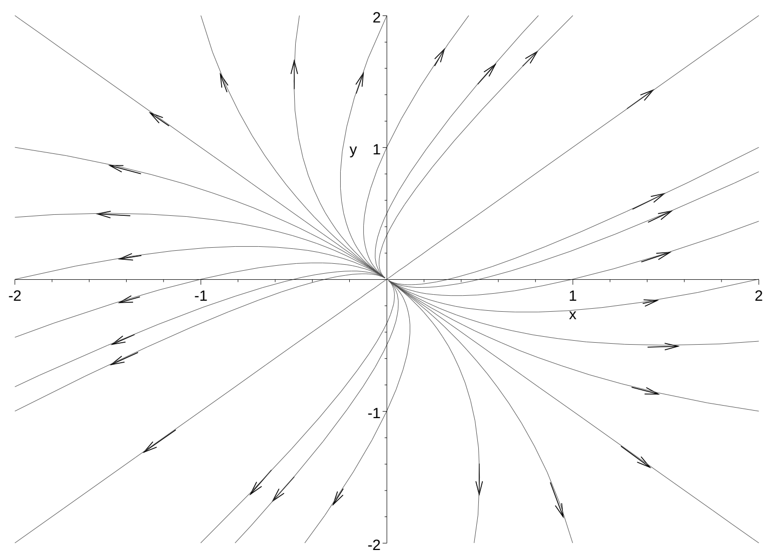
The solution is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}. \tag{5}$$

Sketch the phase diagram and and describe the stationary point.

Solution:

The phase-diagram is



with all the lines going outward. Notice they are all tending towards the same direction as $\mathbf{x}_1$.

3. (4) Find the general solutions for the system

$$\frac{dy_1}{dt} = 2y_1 - y_2 \tag{6}$$
$$\frac{dy_2}{dt} = -4y_2 \tag{7}$$

Sketch the phase diagram and and describe the stationary point.

Solution: So here

$$A = \begin{pmatrix} 2 & -1 \\ 0 & -4 \end{pmatrix}$$

and the spectrum[2] is $\lambda_1 = 2$ corresponding to

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

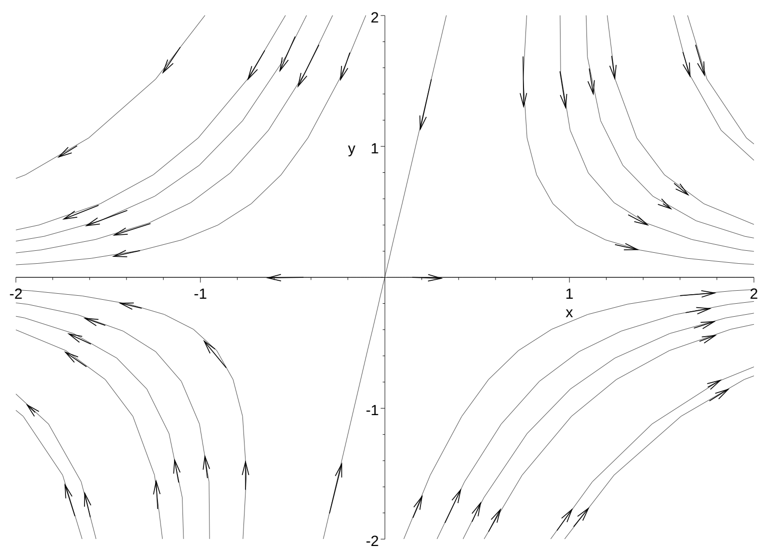
and $\lambda_2 = -4$ corresponding to

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

so the general solution is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 6 \end{pmatrix} e^{-4t}.$$

The phase diagram is



[2]The set of eigenvalues of a matrix is sometimes called its spectrum