2E2 Tutorial Sheet 1, Solutions[1]

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1. (2) Using the linearity of the Laplace transform, calculate the Laplace transform of

$$f(t) = \sinh(at) = \frac{e^{at} - e^{-at}}{2}$$

Solution: Well, just write it out

$$\begin{aligned} \mathcal{L}(\sinh(at)) &= \mathcal{L}\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at}) \\ &= \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a} \\ &= \frac{1}{2}\frac{s+a-(s-a)}{s^2-a^2} = \frac{a}{s^2-a^2} \end{aligned} \quad (1)$$

2. (2) Using the shift theorem find the Laplace transform of

$$f(t) = e^{2t}t^2$$

Solution: Recall the first shift theorem says

$$\mathcal{L}\left(e^{-at}f(t)\right) = F(s-a) \quad (2)$$

where $\mathcal{L}(f) = F(s)$. Now, we know that

$$\mathcal{L}\left(t^2\right) = \frac{2!}{s^3} = \frac{2}{s^3} \quad (3)$$

so, by the shift theorem

$$\mathcal{L}\left(e^{2t}t^2\right) = \frac{2}{(s-2)^3} \quad (4)$$

The next two questions are about the Laplace transform of f', recall the formula

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

3. (2) Find the Laplace transform of both side of the identity

$$\frac{d}{dt}\cosh 3t = 3\sinh 3t$$

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and verify that you get the same answer on each side. The idea is that you do the right hand side using the table entry for $\sinh(3t)$ and the left hand side using the formula for f' with $f = \cosh(3t)$. $\cosh(0) = 1$ by the way.

Solution: We know that

$$\mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}, \qquad \mathcal{L}(\cosh at) = \frac{s}{s^2 - a^2} \tag{5}$$

and $\cosh 0 = 1$ so

$$\begin{aligned}
\mathcal{L}\left(\frac{d}{dt}\cosh 3t\right) &= \mathcal{L}\left(3\sinh 3t\right) \\
s\mathcal{L}\left(\cosh 3t\right) - 1 &= 3\frac{3}{s^2-9} \\
s\frac{s}{s^2-9} - 1 &= 3\frac{3}{s^2-9} \\
s\frac{s}{s^2-9} - \frac{s^2-9}{s^2-9} &= \frac{9}{s^2-9} \\
\frac{9}{s^2-9} &= \frac{9}{s^2-9} \qquad (6)
\end{aligned}$$

4. (2) Find the Laplace transform of both sides of the differential equation

$$2\frac{df}{dt} = 1$$

with initial conditions $f(0) = 4$. By solving the resulting equations find $F(s)$. Based on the Laplace trasforms you know, decide what $f(t)$ is.

Solution: Using linearity of $\mathcal{L}$, plus the property of Laplace transforms of derivatives, we get

$$\begin{aligned}
\mathcal{L}\left(2\frac{dx}{dt}\right) &= \mathcal{L}(1) \\
2\mathcal{L}\left(\frac{df}{dt}\right) &= \frac{1}{s} \\
2sF(s) - 8 &= \frac{1}{s} \qquad (7) \\
& \qquad (8)
\end{aligned}$$

This means that

$$F(s) = \frac{4}{s} + \frac{1}{2s^2} \tag{9}$$

and, since, $\mathcal{L}(t^n) = n!/s^{n+1}$

$$f = 4 + \frac{1}{2}t \tag{10}$$

To verify that this solves the equation note that $f(0) = 4$ as required and $f' = 1/2$.