2E2 Tutorial Sheet 18 Third Term[1]

30 March 2004

Series solution question: (3) The Legendre equation is

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0 \tag{1}$$

and this has series solution with recursion relation

$$a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+2)(n+1)}a_n \tag{2}$$

If l is an integer one of the two series terminates to give a polynomial. This polynomial is called $P_l(x)$ normalized by requiring $P_l(1) = 1$. Write down $P_1(x)$ and $P_3(x)$.

Revision of vectors. The idea in the rest of the sheet is to revise adding vectors, finding their dot products, finding their cross products and working out the scalar triple product.

Adding vectors. Vector are added component by component so if $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ then $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

Length. The length of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^3}$

Dot product. The dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is $|\mathbf{u}||\mathbf{v}|\cos\theta$ where θ is the angle between them. In terms of components $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$

Cross product. The cross product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is a vector with length $|\mathbf{u}||\mathbf{v}|\sin\theta$ where θ is the angle between them. It points perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. In terms of components

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \end{aligned}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the three basis vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.

Scalar triple product of three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

1. (5) For

$$\mathbf{a} = (1, 2, 0), \quad \mathbf{b} = (-3, 2, 0), \quad \mathbf{c} = (2, 3, 4), \quad \mathbf{d} = (6, -7, 2).$$

   calculate (i) $\mathbf{a} + \mathbf{b}$, (ii) $\mathbf{a} \cdot \mathbf{b}$, (iii) $|\mathbf{a}|$, (iv) $\mathbf{a} \times \mathbf{b}$, (v) $\mathbf{b} \times \mathbf{a}$, (vi) $\mathbf{b} \times \mathbf{c}$, (vii) $|\mathbf{a} \times \mathbf{c}|$, (viii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. (ix) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, (x) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

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