2E2 Tutorial Sheet 23 altenative, third term, Solutions[1]

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This problem sheet relates to solving the Dirichlet problem for the heat equation. This is very similar to the Neumann problem we did in class, the difference is in the boundary conditions. Dirichlet boundary conditions fix the derivatives at the end. Consider an iron bar on which heat obeys the heat equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{1}$$

where $T(t,x)$ and the boundary conditions are

$$T(t,0) = T(t,L) = 0 \tag{2}$$

1. (2) Writing $T(t,x) = U(t)X(x)$ show this equation is equivalent to the equations

$$\begin{aligned} \frac{d^2X}{dx^2} &= cX \\ \frac{dU}{dt} &= cU \end{aligned} \tag{3}$$

where c is some constant.

Solution: So, substitute $T = UX$ into the equation. The x differenciation only acts on the X and, in turn, the X only depends on x. In the same way, the t differentiation only acts on U and U in turn only depends on t. Thus, the equation becomes

$$U\frac{d^2X}{dx^2} = \frac{dU}{dt}X \tag{4}$$

Now divide across by UX

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{U}\frac{dU}{dt} \tag{5}$$

Now, the left hand side only depends on x and the right hand side only depends on t. However, x and t are independant variables, they can be varied seperately, so this equation doesn't make sense unless both sides are constant, hence

$$\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= c \\ \frac{1}{U}\frac{dU}{dt} &= c \end{aligned} \tag{6}$$

Multiply the first equation by X and the second by U to get the answer.

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2. (3) Solve these equations and argue from the boundary conditions that c must be negative. Writing $c = -p^2$ calculate what values of p satisfy the boundary conditions.

Solution: Now the U equation is easy to solve, either by guessing the answer and substituting it in, or by integration or indeed by using the Laplace transform, we get

$$U = Ae^{ct} \tag{7}$$

For the second X there are two possibilities, either $c = p^2$ is positive in which case the solution is

$$X = C_1 e^{px} + C_2 e^{-px} \tag{8}$$

or $c = -p^2$ is negative, in which case the solution is

$$X = C_1 \cos px + C_2 \sin px \tag{9}$$

Again, there are a number of ways of seeing these are the solutions, the first is to know they are the solutions and quickly check that differenciating X twive gives the right thing, another is to split the equation into two first order equations and the write in matrix form, as we did after Christmas and the third is to use the Laplace transform.

Now, the positive case can never satisfy the boundary conditions, it gives

$$T = UX = \left(C_1 e^{px} + C_2 e^{-px}\right) e^{p^2 t} \tag{10}$$

where the A has been absorbed into the other two arbitrary constants. and so, if

$$T(t,0) = 0 \tag{11}$$

then

$$(C_1 + C_2)\, e^{p^2 t} \tag{12}$$

so that $C_1 = -C_2$. Now

$$T(t,L) = C_1\left(e^{pL} - e^{-pL}\right) e^{p^2 t} \tag{13}$$

which is only zero if

$$e^{pL} - e^{-pL} = 0 \tag{14}$$

which never happens for non-zero L and p.

Choosing the negative solutions, this gives

$$T = UX = \left[C_1 \cos(px) + C_2 \sin(px)\right] e^{-p^2 t} \tag{15}$$

so

$$T(t,0) = C_1 e^{-p^2 t} \tag{16}$$

and only gives zero when $C_1 = 0$, hence

$$T(t, L) = C_2 \sin{(pL)} e^{-p^2 t} \tag{17}$$

and this is zero if $p = n\pi/L$. Hence, solutions are of the form

$$T = A_n \sin \frac{n\pi x}{L} e^{\frac{n^2\pi^2}{L^2} t} \tag{18}$$

In fact, this is a linear equation so the sum of solutions is a solution and so

$$T = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi x}{L} e^{-\frac{n^2\pi^2}{L^2} t} \tag{19}$$

is a solution.

3. (2) Say the initial condition is

$$T(0, x) = f(x) = 5 \sin \frac{\pi x}{L} - 2 \sin \frac{3\pi x}{L} \tag{20}$$

what is $T(x, t)$.

Solution: So matching to the general solution we have

$$T(t, x) = 5 \sin \frac{\pi x}{L} e^{-\frac{\pi^2}{L^2} t} - 2 \sin \frac{3\pi x}{L} e^{-\frac{9\pi^2}{L^2} t} \tag{21}$$

4. (1) Say the initial condition is

$$T(0, x) = f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L} \tag{22}$$

then what is $T(t, x)$.

Solution: So, mathcing to the general solution we have

$$T = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L} e^{\frac{-n^2\pi^2}{L^2} t} \tag{23}$$

is a solution.