2E2 Tutorial Sheet 23 Third Term[1]

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1. (2) Integrate $\mathbf{F} = \mathbf{i} - y\mathbf{j} + xyz\mathbf{k}$ along $\mathbf{r}(t) = t\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}$ from $(0,0,0)$ to $(1,-1,1)$.

   Solution: So as before the two points correspond to $t = 0$ and $t = 1$ because $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{r}(1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Don't start to think $t_1 = 0$ and $t_2 = 1$ always, this was a coincidence. Now,

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} - 2t\mathbf{j} + \mathbf{k} \tag{1}$$

   so

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 1 + 2ty + xyz \tag{2}$$

   and along the curve $x = t$, $y = -t^2$ and $z = t$. Thus, along the curve

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 1 - 2t^3 - t^4 \tag{3}$$

   and

$$\begin{aligned} \int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} \mathbf{F} \cdot d\mathbf{r} &= \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \\ &= \int_0^1 (1 - 2t^3 - t^4) dt = \frac{3}{10} \end{aligned} \tag{4}$$

2. (2) Integrate $\mathbf{F} = 4x\mathbf{i}$ between $(1,0)$ and $(0,1)$
   (i) Along the quarter circle between those points.
   (ii) Along the straight line.

   Solution: The main difficulty here is in finding the curves. However, for the quarter circle you just need to remember that an equation for a circle of radius one is

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} \tag{5}$$

   to check this, just note that $|\mathbf{r}| = 1$ for all t. Now

$$\frac{d\mathbf{r}}{dt} = -\sin t\mathbf{i} + \cos t\mathbf{j} \tag{6}$$

   and so

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -4x \sin t \tag{7}$$

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Since $x = \cos t$ on the curve this is

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -4\cos t \sin t = -2\sin 2t \tag{8}$$

Finally, not $\mathbf{r}(0) = (1, 0)$ and $\mathbf{r}(\pi/2) = (0, 1)$ so the limits of integration are $t_1 = 0$ and $t_2 = \pi/2$. Thus, the integral is

$$-2\int_0^{\pi/2} \sin 2t dt = \cos 2t]_0^{\pi/2} = -2 \tag{9}$$

As for part (ii) well we need to work out the equation of the line. Using the same reasoning as before,

$$\mathbf{r}(t) = (1-t)\mathbf{i} + t\mathbf{j} \tag{10}$$

with $t_1 = 0$ and $t_2 = 1$. Now,

$$\frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j} \tag{11}$$

and

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -4x \tag{12}$$

Since $x = 1 - t$ on the curve this is

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -4 + 4t \tag{13}$$

Thus, the integral is

$$-4\int_0^1 (1-t)dt = -4t + 2t^2\Big]_0^1 = -2 \tag{14}$$

We got the same answer for (i) and (ii), as we will see, this is because $\mathbf{F}$ has no curl and so its path integral is path independant.

Aside. It is worth noting that another parameterization of the quarter circle is given by

$$\mathbf{r}(t) = \sqrt{1-t^2}\mathbf{i} + t\mathbf{j} \tag{15}$$

with $t_1 = 0$ and $t_2 = 1$. If we had used this instead we would of got

$$\frac{d\mathbf{r}}{dt} = -\frac{t}{\sqrt{1-t^2}}\mathbf{i} + \mathbf{j} \tag{16}$$

and

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -4x\frac{t}{\sqrt{1-t^2}} = -4t \tag{17}$$

along the curve so the integral would be

$$-4\int_0^1 t dt = -2t^2\Big]_0^1 = -2 \tag{18}$$

which is the same.

3. (2) Integrate $\oint_C (xdy - ydx)$ where C is the circle $x^2 + y^2 = 1$.

   The way to do this problem is to first rewrite it using Green's theore m, the resulting two-dimensional integral is easier than the line integral.

   Solution: Well $\oint_C (xdy - ydx)$ is a line integral of the form appearing in the Green's theorem with $F_1 = -y$ and $F_2 = x$. Appling Greens theorem we get

$$\int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = 2\int\int_R dxdy \tag{19}$$

   and the area of the disk is π so the answer is 2π.

4. (2) Use Green's theorem on the plane to integrate $\oint_C (F_1 dx + F_2 dy)$ where $F = (y, x)$ and C is a circle of radius 2 around the origin.

   As before, question 3 is a lot easier than it might look, find the cor responding two-dimensional integral using Green's theorem and note that it is trivial.

   Solution: Here,

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \tag{20}$$

   so by Green's theorem the line integral vanishes.