Method of Frobenius, example and problem[1]

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Consider the equation

$$ty' + (1-t)y = 0 \tag{1}$$

We encounter a problem if we try to solve this by the usual series solution method, that is, by assuming there is a solution of the form

$$y = \sum_{n=0}^{\infty} a_n t^n \tag{2}$$

Let us try it this way, see what goes wrong and then see why the method of Fobenius avoids that problem. First write the various term of the equation in terms of the sum

$$ty' = \sum_{n=0}^{\infty} n a_n t^n, \qquad ty = \sum_{n=0}^{\infty} a_n t^{n+1} \tag{3}$$

Substituting into the equation we get

$$\sum_{n=0}^{\infty} n a_n t^n + \sum_{n=0}^{\infty} a_n t^n - \sum_{n=0}^{\infty} a_n t^{n+1} = 0 \tag{4}$$

and so we have to shift the index in the first two terms up to get t^{n+1}:

$$\sum_{n=-1}^{\infty} (n+1) a_{n+1} t^{n+1} + \sum_{n=-1}^{\infty} a_{n+1} t^{n+1} - \sum_{n=0}^{\infty} a_n t^{n+1} = 0 \tag{5}$$

and then taking terms out of the sum so that all the sums are over the same range gives

$$a_0 + \sum_{n=0}^{\infty} [(n+2)a_{n+1} - a_n] t^{n+1} = 0 \tag{6}$$

so the recursion relation is

$$a_{n+1} = \frac{1}{n+2} a_n \tag{7}$$

along with the condition $a_0 = 0$. This is the problem, we have a one-step recusion so if a_0 is zero so is a_1 and if $a_1 = 0$ so is a_2 and so on for ever.

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The point is that the equation does have a solution but not of the normal form. To see this we make a weaker assumuption about the solution and look for a series solution of the form

$$y = \sum_{n=0}^{\infty} a_n t^{n+r} = t^r \sum_{n=0}^{\infty} a_n t^n \tag{8}$$

where, as we will see, r is some constant which will be determined. Now, we do all the same stuff as before

$$ty' = \sum_{n=0}^{\infty} (n+r) a_n t^{n+r}, \qquad ty = \sum_{n=0}^{\infty} a_n t^{n+r+1} \tag{9}$$

giving

$$\sum_{n=0}^{\infty} (n+r) a_n t^{n+r} + \sum_{n=0}^{\infty} a_n t^{n+r} - \sum_{n=0}^{\infty} a_n t^{n+r+1} = 0 \tag{10}$$

and hence

$$\sum_{n=-1}^{\infty} (n+r+1) a_{n+1} t^{n+r+1} + \sum_{n=-1}^{\infty} a_{n+1} t^{n+r+1} - \sum_{n=0}^{\infty} a_n t^{n+r+1} = 0 \tag{11}$$

leading to

$$(1+r)a_0 + \sum_{n=0}^{\infty} [(n+r+2)a_{n+1} - a_n] t^{n+r+1} = 0 \tag{12}$$

so now the recursion relation is

$$a_{n+1} = \frac{1}{n+r+2} a_n \tag{13}$$

with $(1+r)a_0 = 0$. This looks like just another equation for a_0, but, if $r = -1$ it is satisfied for any a_0, we don't need $a_0 = 0$ anymore. Good, so the series solution is

$$y = \frac{1}{t} \sum_{n=0}^{\infty} a_n t^n \tag{14}$$

with

$$a_{n+1} = \frac{1}{n+1} a_n \tag{15}$$

and a_0 arbitrary. The first few non-zero terms are

$$y = a_0 \left(\frac{1}{t} + 1 + \frac{1}{2} t + \frac{1}{6} t^2 + \dots \right) \tag{16}$$

Here is an example to try yourself

$$ty'' + 2y' + y = 0 \tag{17}$$

Notice that this is a second order equation so that there should be two solutions. If you don't use the method of Frobenius you appear to get only one solution. In this way it is a small bit different from the example above where we expect one solution and get none unless we use the method of Frobenius.