Method of Frobenius, solution of problem[1]

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So the equation we want to solve is

$$ty'' + 2y' + y = 0 \tag{1}$$

and to solve it by the method of Frobenius we assume the solution has the form

$$y = \sum_{n=0}^{\infty} a_n t^{n+r} \tag{2}$$

Now, work out the various terms

$$ty'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r-1} \tag{3}$$

gives

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r-1} + 2\sum_{n=0}^{\infty} (n+r)a_n t^{n+r-1} + \sum_{n=0}^{\infty} a_n t^{n+r} = 0. \tag{4}$$

Shifting the indices in the first two terms upwards, leaving out the intermediate steps with $m = n - 1$ and then renaming m as n, gives

$$\sum_{n=-1}^{\infty} (n+r+1)(n+r)a_{n+1} t^{n+r} + 2\sum_{n=-1}^{\infty} (n+r+1)a_{n+1} t^{n+r} + \sum_{n=0}^{\infty} a_n t^{n+r} = 0. \tag{5}$$

and so

$$r(r-1)a_0 + 2ra_0 + \sum_{n=0}^{\infty} [(n+r+1)(n+r)a_{n+1} + 2(n+r+1)a_{n+1} + a_n]t^{nq+r} = 0. \tag{6}$$

so

$$r(r+1)a_0 = 0 \tag{7}$$

and

$$a_{n+1} = -\frac{1}{(n+r+2)(n+r+1)} a_n \tag{8}$$

Thus, we can avoid $a_0 = 0$ by setting $r = 0$ or $r = -1$. This means that there are two solutions, a $r = 0$ solution and a $r = -1$ solution. Since we have a one-step recursion relation this is good because a_1 isn't arbitrary as it usually is for second order equations. Instead of having arbitrary a_0 and a_1 the solution is the sum of the $r = 0$ series and the $r = -1$ series with an arbitrary a_0 for each series. Not using the method of Frobenius would only have given the $r = 0$ solution.

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