Some Laplace transform examples with complex numbers[1]

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We have seen that using Laplace transforms to solve differential equations usually comes down to a partial fraction expansion. For some differential equation this expansion will involve complex numbers. This not only complicates find the partial fraction expansion, it also causes difficulty when you try to turn a solution which looks complex but is actually real, into one that looks real.

Consider the example of trying to solve

$$f'' + 2f' + 5 = 1 \tag{1}$$

with $f(0) = f'(0) = 0$. Taking the Laplce transform of each side we get

$$(s^2 + 2s + 5)F = \frac{1}{s} \tag{2}$$

Hence,

$$F = \frac{1}{s(s^2 + 2s + 5)} \tag{3}$$

We now need to factorize $s^2 + 2s + 5$. Solving using the formula gives

$$s = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i \tag{4}$$

so

$$s^2 + 2s + 5 = (s + 1 + 2i)(s + 1 - 2i) \tag{5}$$

Next, partial fractions. This part is no different from the examples without complex numbers, but it is trickier.

$$\frac{1}{s(s + 1 - 2i)(s + 1 + 2i)} = \frac{A}{s} + \frac{B}{s + 1 - 2i} + \frac{C}{s + 1 + 2i} \tag{6}$$

Multiplying across by the denominator gives

$$1 = A(s + 1 - 2i)(s + 1 + 2i) + Bs(s + 1 + 2i) + Cs(s + 1 - 2i) \tag{7}$$

Choosing $s = 0$ gives

$$1 = 5A \tag{8}$$

and hence $A = 1/5$. Next, $s = -1 + 2i$ gives

$$1 = B(-1 + 2i)(4i) = -8 - 4i \tag{9}$$

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so

$$B = -\frac{1}{8+4i} \tag{10}$$

Finally, $s = -1 - 2i$ gives

$$1 = C(-1-2i)(-4i) = -8 + 4i \tag{11}$$

giving

$$C = -\frac{1}{8-4i} \tag{12}$$

Putting all this together we get

$$F = \frac{1}{5s} - \frac{1}{8+4i}\frac{1}{s+1-2i} - \frac{1}{8-4i}\frac{1}{s+1+2i} \tag{13}$$

and, using the tables, this gives

$$f = \frac{1}{5} - \frac{1}{8+4i}e^{-(1-2i)t} - \frac{1}{8-4i}e^{-(1+2i)t} \tag{14}$$

Although this does tell us what f is, it does it in a complicated way. For a start, this makes it look like f is complex, when we know that f satisfies a real differential equation and should be real. To rewrite this in a real form we need to do two things, we change the fractions with complex numbers on the bottom to fractions with complex numbers on the top and we expand the exponentials using the formula

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned} \tag{15}$$

Note that the second of these formulas follows from the first using

$$\begin{aligned} \cos(-\theta) &= \cos\theta \\ sin(-\theta) &= -\sin\theta \end{aligned} \tag{16}$$

First the fractions, remember there is a standard method for dividing by a complex number: you multiply above and below by the conjugate. Hence

$$\frac{1}{8+4i} = \frac{1}{8+4i}\frac{8-4i}{8-4i} \tag{17}$$

which makes sense because the second fraction is equal to one. Now

$$(8+4i)(8-4i) = 8^2 - (4i)^2 = 64 + 16 = 80 \tag{18}$$

and so

$$\frac{1}{8+4i} = \frac{8-4i}{80} = \frac{2-i}{20} \tag{19}$$

You can do the same with the other complex fraction, it is quicker just to note it is the same as the one we just did except the sign of i is different, so,

$$\frac{1}{8+4i} = \frac{8+4i}{80} = \frac{2+i}{20} \tag{20}$$

Now we have

$$f = \frac{1}{5} - \frac{2-i}{20}e^{-(1-2i)t} - \frac{2+i}{20}e^{-(1+2i)t} \tag{21}$$

Next we use the fact

$$\begin{aligned} e^{-(1-2i)t} &= e^{-t}e^{2it} \\ e^{-(1+2i)t} &= e^{-t}e^{-2it} \end{aligned} \tag{22}$$

and so

$$f = \frac{1}{5} - \frac{1}{20}\left[(2-i)e^{2it} - (2+i)e^{-2it}\right]e^{-t} \tag{23}$$

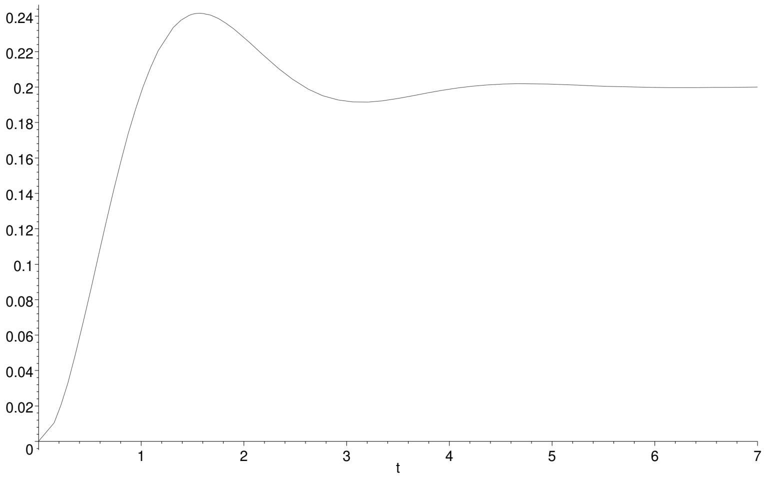
Next, writing the imaginary exponentials out, we find

$$f = \frac{1}{5} - \frac{1}{20}\left[(2-i)(\cos 2t + i\sin 2t) - (2+i)(\cos 2t + i\sin 2t)\right]e^{-t} \tag{24}$$

Multiply out the brackets, all the i bits cancel and we are left with

$$f = \frac{1}{5} - \frac{1}{10}(2\cos 2t + \sin 2t)e^{-t} \tag{25}$$

This is what it looks like



Here is another example. Say you wanted to use the Laplace transform to solve the equation:

$$\frac{d^2 f}{dt^2} + 2\frac{df}{dt} + 2f = 1 \tag{26}$$

with initial conditions $f(0) = \dfrac{df}{dt}(0) = 0$. Taking the Laplace transform of both sides we get.

$$(s^2 + 2s + 2)F = \frac{1}{s} \tag{27}$$

Hence,

$$F(s) = \frac{1}{s(s^2 + 2s + 2)} \tag{28}$$

and we can factor the denominator using the roots $s = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{-4}}{2} = -1 \pm \sqrt{-1} = -1 \pm i$ of the quadratic. So complex partial fractions take the form

$$\begin{aligned}
F &= \frac{1}{s(s-(-1+i))(s-(-1-i))} = \frac{A}{s} + \frac{B}{s+1-i} + \frac{C}{s+1+i} \\
1 &= A(s+1-i)(s+1+i) + Bs(s+1+i) + Cs(s+1-i) \\
&= A(s^2+2s+2) + Bs(s+1+i) + Cs(s+1-i) \\
\underline{s=0:} \qquad 1 &= 2A \\
A &= \frac{1}{2} \\
\underline{s=-1+i:} \qquad 1 &= 0 + B(-1+i)(2i) + 0 = 2B(-i-1) = -2(1+i)B \\
B &= \frac{1}{-2(1+i)} = -\frac{1-i}{2(1+i)(1-i)} \\
&= -\frac{1-i}{4} = -\frac{1}{4} + \frac{i}{4} \\
\underline{s=-1-i:} \qquad 1 &= 0 + 0 + C(-1-i)(-2i) + 0 = 2C(i-1) = 2(-1+i)C \\
C &= \frac{1}{2(-1+i)} = \frac{-1-i}{2(-1+i)(-1-i)} \\
&= -\frac{1+i}{4} = -\frac{1}{4} - \frac{i}{4} \\
&= \overline{B} \\
F &= \frac{1}{2}\frac{1}{s} + \left(-\frac{1}{4} + \frac{i}{4}\right)\frac{1}{s+1-i} + \left(-\frac{1}{4} - \frac{i}{4}\right)\frac{1}{s+1+i} \qquad (29)
\end{aligned}$$

Via the tables, we have

$$\begin{aligned}
f(t) &= \frac{1}{2} + \left(-\frac{1}{4} + \frac{i}{4}\right)e^{(-1+i)t} + \left(-\frac{1}{4} - \frac{i}{4}\right)e^{(-1-i)t} \\
&= \frac{1}{2} + \left(-\frac{1}{4} + \frac{i}{4}\right)e^{-t}e^{-it} + \left(-\frac{1}{4} - \frac{i}{4}\right)e^{-t}e^{-it} \\
&= \frac{1}{2} + \left(-\frac{1}{4} + \frac{i}{4}\right)e^{-t}(\cos t + i\sin t) + \left(-\frac{1}{4} - \frac{i}{4}\right)e^{-t}(\cos t - i\sin t) \\
&= \frac{1}{2} + \frac{e^{-t}}{4}(-\cos t - \sin t - i\sin t + i\cos t) + \frac{e^{-t}}{4}(-\cos t - \sin t + i\sin t - i\cos t) \\
&= \frac{1}{2} - \frac{e^{-t}}{2}(\cos t + \sin t) \qquad (30)
\end{aligned}$$