Simple example with Heaviside function[1]

6 November 2001, revised 8 November 2004

This is a somple example involving a differential equation with a Heaviside function. I have included it so that I could show you a plot of the solution, the effect of the Heaviside function is clearly visible. Consider

$$f' - f = H_1(t) \tag{1}$$

with $f(0) = 1$. Taking the Laplace transform of the equation gives

$$(s-1)F = 1 + \frac{e^{-s}}{s} \tag{2}$$

and so

$$F = \frac{1}{s-1} + \frac{e^{-s}}{s(s-1)} = \frac{1}{s-1} + \left(-\frac{1}{s} + \frac{1}{s-1}\right)e^{-s} \tag{3}$$

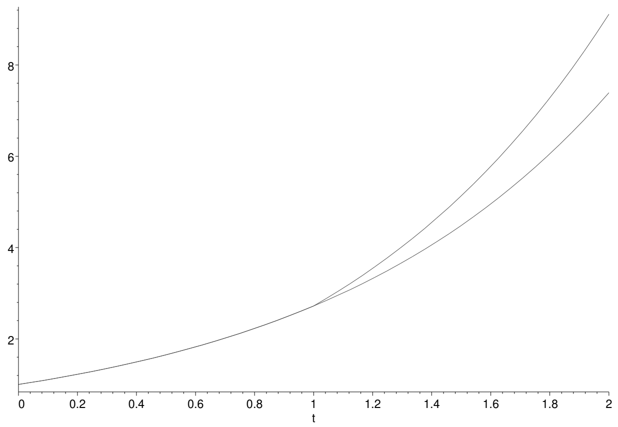
Since

$$\mathcal{L}\left(-1 + e^t\right) = -\frac{1}{s} + \frac{1}{s-1} \tag{4}$$

we can use the third shift theorem to find that

$$f = e^t + H_1(t)\left(-1 + e^{t-1}\right) \tag{5}$$

In the graph I have plotted this f along with the graph of e^t on its own. The upper of the two lines is f, the lower is e^t. Until $t = 1$ they are the same, then the Heaviside function in f switches on. You can see that the two are the same until $t = 1$ and after that f is larger than e^t. The thing to notice is that f is not discontinuous, but it does change its behaviour and its derivative is discontinuous at this point.



[1] Conor Houghton, `houghton@maths.tcd.ie` please send me any corrections.