More plotting phase diagrams [1]

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These used to be problem sheets question, the first I still use, but the second wasn't asked this year, it is basically the same as the first but with a wind resistance term added.

1. By linearizing around the critical points, draw the phase plane portrait of

$$y'' = \cos 2y \tag{1}$$

Solution: As before, rewrite as a first order system:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= \cos 2y_1 \end{aligned} \tag{2}$$

now, the critical points are located where $y_1' = y_2' = 0$. This happens when $y_2 = 0$ and $\cos 2y_1 = 0$, that means $2y_1 = n\pi/2$ where n is an odd integer, or $y_1 = n\pi/4$ where again n is an odd integer.

Near $y_1 = \pi/4$ write $y_1 = \pi/4 + \eta$ and use $\cos 2y_1 = \cos 2(\pi/4 + eta) = -\sin 2\eta$ and this linearizes as $\sin 2\eta \sim 2\eta$ so the system becomes

$$\begin{aligned} \eta' &= y_2 \\ y_2' &= -2\eta. \end{aligned} \tag{3}$$

This is a center. The matrix is

$$A = \left(\begin{array}{cc} 0 & 1 \\ -2 & 0 \end{array} \right) \tag{4}$$

and so, by calculating the eigenvalues and eigenvectors, the general solution is

$$\left(\begin{array}{c} \eta \\ y_2 \end{array} \right) = c_1 \left(\begin{array}{c} 1 \\ \sqrt{2}i \end{array} \right) e^{\sqrt{2}it} + c_2 \left(\begin{array}{c} 1 \\ -\sqrt{2}i \end{array} \right) e^{-\sqrt{2}it} \tag{5}$$

and by beginning at $\eta = r$ and $y_2 = 0$ we get

$$\left(\begin{array}{c} \eta \\ y_2 \end{array} \right) = r \left(\begin{array}{c} \cos\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t \end{array} \right) \tag{6}$$

so the saddle point is an ellipse with the vertical $\sqrt{2}$ times as long as the horizontal.

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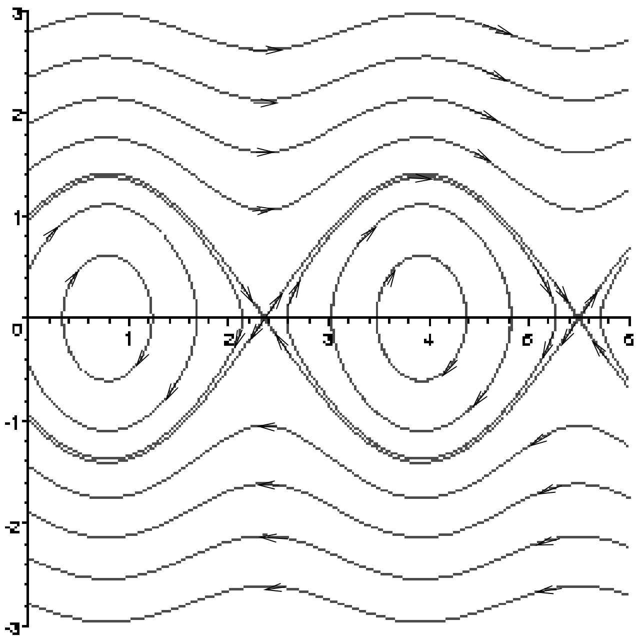
Near $y_1 = 3\pi/4$ write $y_1 = 3\pi/4 + \eta$ and use $\cos 2y_1 = \cos 2(3\pi/4 + eta) = \sin 2\eta$ and this linearizes as $\sin 2\eta \sim 2\eta$ so the system becomes

$$\begin{aligned} \eta' &= y_2 \\ y_2' &= 2\eta \end{aligned} \tag{7}$$

This is a saddle-point with eigenvalues $\pm\sqrt{2}$ and eigenvectors

$$\begin{pmatrix} 1 \\ \pm\sqrt{2} \end{pmatrix}. \tag{8}$$

This pattern repeats by periodicity, the phase portrait is



2. How is this changed if we add a first order term

$$y'' = -\frac{1}{2}y' + \cos 2y \tag{9}$$

Draw the phase portrait in this case.

Solution: When the first order term has been added the system becomes

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{1}{2}y_2 + \cos 2y_1 \end{aligned} \tag{10}$$

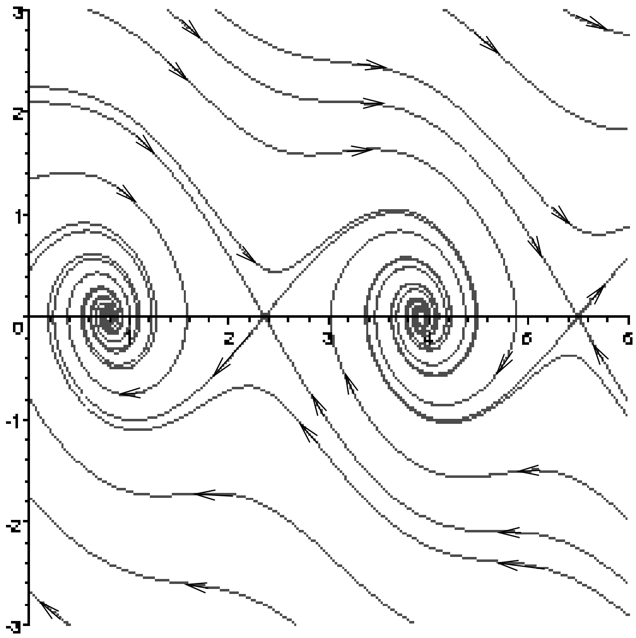
The critical points are in the same place, but the eigenvalues and eigenvectors are changed, near $\pi/4$ we get

$$\begin{aligned} \eta' &= y_2 \\ y_2' &= -\frac{1}{2}y_2 - 2\eta \end{aligned} \tag{11}$$

The matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -1/2 \end{pmatrix} \tag{12}$$

and has eigenvalues $-1/4 \pm \sqrt{31}i/4$ and so it is an inward moving spiral. In the same way, the saddle-point is still a saddle-point but with slightly different eigenvalues. The phase portrait is



Here, to answer the question, you just need to say that the ellipsis becomes an inward moving spiral because it gets a negative real part and the saddle point eigenvectors are changed slightly and then draw the phase portrait very roughly.