A long example of solving a differential equation using Laplace Transform[1]

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Consider

$$y'' - 4y' + 3y = 6t - 8 \tag{1}$$

with intial conditions $y(0) = y'(0) = 0$. If we write $Y = \mathcal{L}(y)$ the Laplace transform is

$$\begin{aligned}
s^2Y - 4sY + 3Y &= \frac{6}{s^2} - \frac{8}{s} \\
(s^2 - 4s + 3)Y &= \frac{6}{s^2} - \frac{8}{s} \\
Y &= \frac{6}{s^2(s^2 - 4s + 3)} - \frac{8}{s(s^2 - 4s + 3)} \qquad (2)
\end{aligned}$$

Now we have to put this into a form which allows us to take the inverse transform. The second term isn't so bad. Since $s^2 - 4s + 3 = (s - 1)(s - 3)$ we write

$$\begin{aligned}
\frac{1}{s(s-1)(s-3)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-3} \\
1 &= (s-1)(s-3)A + s(s-3)B + s(s-1)C \qquad (3)
\end{aligned}$$

Thus, choosing $s = 0$ gives $A = 1/3$, $s = 1$ gives $B = -1/2$ and choosing $s = 3$ gives $C = 1/6$. Thus

$$\frac{1}{s(s^2 - 4s + 3)} = \frac{1}{3s} - \frac{1}{2(s-1)} + \frac{1}{6(s-3)} \tag{4}$$

The other expansion is harder because it has a repeated root: in

$$\frac{1}{s^2(s-1)(s-3)} \tag{5}$$

the s factor appears as a square. To deal with this you have to include a $1/s$ term and a $1/s^2$ term in the partial fraction expansion.

$$\begin{aligned}
\frac{1}{s^2(s-1)(s-3)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-3} \\
1 &= s(s-1)(s-3)A + (s-1)(s-3)B + s^2(s-3)C + s^2(s-1)D \qquad (6)
\end{aligned}$$

Now taking $s = 0$ gives $B = 1/3$, $s = 1$ gives $C = -1/2$ and $s = 3$ gives $D = 1/18$. There is no convenient choice of s that gives A on its own, so we just substitute in any other value, $s = 2$ say and by putting in the values of B, C and D we get

$$1 = -2A - \frac{1}{3} + 2 + \frac{2}{9} \tag{7}$$

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and hence $A = 4/9$. Thus

$$\frac{1}{s^2(s-1)(s-3)} = \frac{4}{9s} + \frac{1}{3s^2} - \frac{1}{2(s-1)} + \frac{1}{18(s-3)} \tag{8}$$

Now we can put everything together

$$\begin{aligned}
Y &= 6\left(\frac{4}{9s} + \frac{1}{3s^2} - \frac{1}{2(s-1)} + \frac{1}{18(s-3)}\right) \\
&\quad -8\left(\frac{1}{3s} - \frac{1}{2(s-1)} + \frac{1}{6(s-3)}\right) \qquad (9)
\end{aligned}$$

and if we do the algebra we find

$$Y = \frac{2}{s^2} + \frac{1}{s-1} - \frac{1}{s-3} \tag{10}$$

which means that

$$y = 2t + e^t - e^{3t} \tag{11}$$

and you can check that this does solve the original equation.