A long example of solving a differential equation using Laplace Transform\(^1\)

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Consider

\[ y'' - 4y' + 3y = 6t - 8 \] \hspace{1cm} (1)

with initial conditions \( y(0) = y'(0) = 0 \). If we write \( Y = \mathcal{L}(y) \) the Laplace transform is

\[
\begin{align*}
    s^2Y - 4sY + 3Y & = \frac{6}{s} - \frac{8}{s} \\
    (s^2 - 4s + 3)Y & = \frac{6}{s} - \frac{8}{s} \\
    Y & = \frac{6}{s^2(s^2 - 4s + 3)} - \frac{8}{s(s^2 - 4s + 3)}
\end{align*}
\] \hspace{1cm} (2)

Now we have to put this into a form which allows us to take the inverse transform. The second term isn’t so bad. Since \( s^2 - 4s + 3 = (s - 1)(s - 3) \) we write

\[
\frac{1}{s(s - 1)(s - 3)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s - 3}
\]

Thus, choosing \( s = 0 \) gives \( A = 1/3 \), \( s = 1 \) gives \( B = -1/2 \) and choosing \( s = 3 \) gives \( C = 1/6 \). Thus

\[
\frac{1}{s(s^2 - 4s + 3)} = \frac{1}{3s} - \frac{1}{2(s - 1)} + \frac{1}{6(s - 3)}
\] \hspace{1cm} (4)

The other expansion is harder because it has a repeated root: in

\[
\frac{1}{s^2(s^2 - 4s + 3)}
\]

the \( s \) factor appears as a square. To deal with this you have to include a \( 1/s \) term and a \( 1/s^2 \) term in the partial fraction expansion.

\[
\frac{1}{s^2(s - 1)(s - 3)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s - 3} + \frac{D}{s^2}
\]

Now taking \( s = 0 \) gives \( B = 1/3 \), \( s = 1 \) gives \( C = -1/2 \) and \( s = 3 \) gives \( D = 1/18 \). There is no convenient choice of \( s \) that gives \( A \) on its own, so we just substitute in any other value, \( s = 2 \) say and by putting in the values of \( B, C \) and \( D \) we get

\[
1 = -2A - \frac{1}{3} + 2 + \frac{2}{9}
\] \hspace{1cm} (7)

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and hence $A = 4/9$. Thus

$$\frac{1}{s^2(s-1)(s-3)} = \frac{4}{9s} + \frac{1}{3s^2} - \frac{1}{2(s-1)} + \frac{1}{18(s-3)}$$

(8)

Now we can put everything together

$$Y = 6 \left( \frac{4}{9s} + \frac{1}{3s^2} - \frac{1}{2(s-1)} + \frac{1}{18(s-3)} \right) - 8 \left( \frac{1}{3s} - \frac{1}{2(s-1)} + \frac{1}{6(s-3)} \right)$$

(9)

and if we do the algebra we find

$$Y = \frac{2}{s^2} + \frac{1}{s-1} - \frac{1}{s-3}$$

(10)

which means that

$$y = 2t + e^t - e^{3t}$$

(11)

and you can check that this does solve the original equation.