## A long example of solving a differential equation using Laplace Transform<sup>1</sup>

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Consider

$$y'' - 4y' + 3y = 6t - 8 \tag{1}$$

with initial conditions y(0) = y'(0) = 0. If we write  $Y = \mathcal{L}(y)$  the Laplace transform is

$$s^{2}Y - 4sY + 3Y = \frac{6}{s^{2}} - \frac{8}{s}$$

$$(s^{2} - 4s + 3)Y = \frac{6}{s^{2}} - \frac{8}{s}$$

$$Y = \frac{6}{s^{2}(s^{2} - 4s + 3)} - \frac{8}{s(s^{2} - 4s + 3)}$$
(2)

Now we have to put this into a form which allows us to take the inverse transform. The second term isn't so bad. Since  $s^2 - 4s + 3 = (s - 1)(s - 3)$  we write

$$\frac{1}{s(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-3}$$
  
1 = (s-1)(s-3)A + s(s-3)B + s(s-1)C (3)

Thus, choosing s = 0 gives A = 1/3, s = 1 gives B = -1/2 and choosing s = 3 gives C = 1/6. Thus

$$\frac{1}{s(s^2 - 4s + 3)} = \frac{1}{3s} - \frac{1}{2(s - 1)} + \frac{1}{6(s - 3)}$$
(4)

The other expansion is harder because it has a repeated root: in

$$\frac{1}{s^2(s-1)(s-3)}$$
(5)

the s factor appears as a square. To deal with this you have to include a 1/s term and a  $1/s^2$  term in the partial fraction expansion.

$$\frac{1}{s^2(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-3}$$
  
1 =  $s(s-1)(s-3)A + (s-1)(s-3)B + s^2(s-3)C + s^2(s-1)D(6)$ 

Now taking s = 0 gives B = 1/3, s = 1 gives C = -1/2 and s = 3 gives D = 1/18. There is no convenient choice of s that gives A on its own, so we just substitute in any other value, s = 2 say and by putting in the values of B, C and D we get

$$1 = -2A - \frac{1}{3} + 2 + \frac{2}{9} \tag{7}$$

<sup>&</sup>lt;sup>1</sup>Conor Houghton, houghton@maths.tcd.ie please send me any corrections.

and hence A = 4/9. Thus

$$\frac{1}{s^2(s-1)(s-3)} = \frac{4}{9s} + \frac{1}{3s^2} - \frac{1}{2(s-1)} + \frac{1}{18(s-3)}$$
(8)

Now we can put everything together

$$Y = 6\left(\frac{4}{9s} + \frac{1}{3s^2} - \frac{1}{2(s-1)} + \frac{1}{18(s-3)}\right) - 8\left(\frac{1}{3s} - \frac{1}{2(s-1)} + \frac{1}{6(s-3)}\right)$$
(9)

and if we do the algebra we find

$$Y = \frac{2}{s^2} + \frac{1}{s-1} - \frac{1}{s-3} \tag{10}$$

which means that

$$y = 2t + e^t - e^{3t} (11)$$

and you can check that this does solve the original equation.