Another example with Heaviside function and with the Dirac delta[1]

18 November 2001

So, consider the differential equation:

$$f'' + f' - 6f = \begin{cases} 0 & t < 3 \\ 2 & 3 \leq t < 5 \\ 0 & 5 \leq t \end{cases} \tag{1}$$

with $f(0) = f'(0) = 0$. The first thing to do is to rewrite the right hand side in terms of the Heaviside function. The key point is that the Heaviside function $H_a(t)$ is zero until you get to a and then it is one after that. Now the function on the right hand side is zero until we get to three and then it is two, so it behaves like $2H_3(t)$, however, at five it goes back down to zero, so we have to take away $2H_5(t)$, in short:

$$2H_3(t) - 2H_5(t) = \begin{cases} 0 & t < 3 \\ 2 & 3 \leq t < 5 \\ 0 & 5 \leq t \end{cases} \tag{2}$$

The first Heaviside function switches on at $t = 3$ and brings you up to two, the second switches on at five and brings you back down to zero.

Now, the differential equation is

$$f'' + f' - 6f = 2H_3(t) - 2H_5(t) \tag{3}$$

with $f(0) = f'(0) = 0$ and we take the Laplace transform of both sides:

$$s^2F + sF - 6F = \frac{2}{s}\left(e^{-3s} - e^{-5s}\right) \tag{4}$$

or

$$F = \frac{2}{s(s+3)(s-2)}\left(e^{-3s} - e^{-5s}\right) \tag{5}$$

Now, partial fractions has

$$\frac{2}{s(s+3)(s-2)} = -\frac{1}{3s} + \frac{2}{15}\frac{1}{s+3} + \frac{1}{5}\frac{1}{s-2} \tag{6}$$

Now,

$$-\frac{1}{3s} + \frac{2}{15}\frac{1}{s+3} + \frac{1}{5}\frac{1}{s-2} = \mathcal{L}\left(-\frac{1}{3} + \frac{2}{15}e^{-3t} + \frac{1}{5}e^{2t}\right) \tag{7}$$

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Now, in the expression for F this gets multiplied by various exponential factors, the effect of this is to delay the answer:

$$f=\left(-\frac{1}{3}+\frac{2}{15}e^{-3t+9}+\frac{1}{5}e^{2t-6}\right)H_3(t)+\left(-\frac{1}{3}+\frac{2}{15}e^{-3t+15}+\frac{1}{5}e^{2t-10}\right)H_5(t) \tag{8}$$

I tried plotting the answer for this, but it is hard to see any change at $t=5$, the system has started growing at this point and switching off the input doesn't have a big enough effect to be obvious in a plot.

Now, here is a similar problem, but with a Dirac delta function:

$$f''+f'-6f=\delta(t-4) \tag{9}$$

with $f(0)=f'(0)=0$. Using $\mathcal{L}[\delta(t-a)]=e^{as}$ this gives

$$s^2F+sF-6F=e^{4s} \tag{10}$$

or

$$F=\frac{e^{4s}}{(s+3)(s-2)} \tag{11}$$

By partial fractions we have

$$\frac{1}{(s+3)(s-2)}=\frac{1}{5}\frac{1}{s-2}-\frac{1}{5}\frac{1}{s+3}=\mathcal{L}\left(\frac{1}{5}e^{2t}-\frac{1}{5}e^{-3t}\right) \tag{12}$$

so, the e^{4s} causes a delay of four and we have

$$f=\left(\frac{1}{5}e^{2t-8}-\frac{1}{5}e^{-3t+12}\right)H_4(t) \tag{13}$$