Examples of inward improper nodes [1]

18 January 2003

In class on the 17 January 2003 I considered drawing the phase diagram for the following system

$$\begin{aligned} y_1' &= -5y_1 - y_2 \\ y_2' &= 12y_1 + 2y_2 \end{aligned} \tag{1}$$

As we will see, both the eigenvalues for this system are negative and so it is an example of an improper node. It is a slightly awkward example to draw, again, as we shall see, the angle between the two eigenvectors is very small. In this note I will solve this system and plot the phase diagram and then do the same thing for a nicer example.

Converting the system into matrix form gives

$$\mathbf{y} = \begin{pmatrix} -5 & -1 \\ 12 & 2 \end{pmatrix} \mathbf{y} \tag{2}$$

Next we look for the eigenvectors and eigenvalues. The eigenvalues solve the equation

$$\mathrm{d}et \begin{pmatrix} -5-\lambda & -1 \\ 12 & 2-\lambda \end{pmatrix} = 0 \tag{3}$$

so

$$(-5-\lambda)(2-\lambda) + 12 = 0 \tag{4}$$

or

$$\lambda^2 + 3\lambda + 2 = 0 \tag{5}$$

which means that λ is -1 or -2, let $\lambda_1 = -1$ and $\lambda_2 = -2$.

Next, we need the eigenvectors, first for λ_1,

$$\begin{pmatrix} -5 & -1 \\ 12 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = - \begin{pmatrix} a \\ b \end{pmatrix} \tag{6}$$

so $-5a - b = -a$ or $4a = -b$ so an example is

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \tag{7}$$

Now, for λ_2

$$\begin{pmatrix} -5 & -1 \\ 12 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -2 \begin{pmatrix} a \\ b \end{pmatrix} \tag{8}$$

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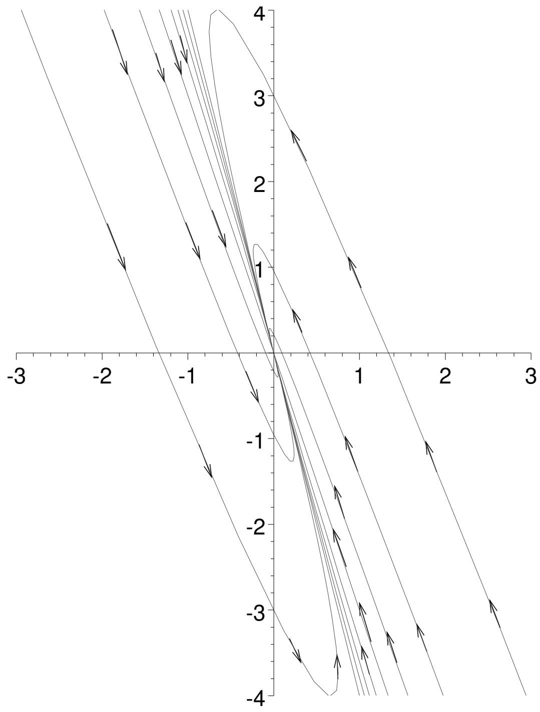
so $-5a - b = -2a$ or $3a = -b$ so an example is

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{9}$$

We can immeadiately write down the general solution

$$bfy = C_1 e^{-t}\mathbf{x}_1 + C_2 e^{-2t}\mathbf{x}_2 \tag{10}$$

so, no matter where we start, no matter what C_1 and C_2 are, y_1 and y_2 get small as t increases because the exponentials get small. The distance in the $\mathbf{x}_2$ direction gets smaller faster because the exponential multiplying it get smaller faster. Unfortunately that might not be so easy to see in the phase diagram because the two eigenvalues are nearly in the same direction:



As you can see, all the trajectories move inwards. The $\mathbf{x}_1$-direction is getting small slower than the $\mathbf{x}_2$-direction, this is what causes the lines to curve around the way they do. The thing to bear in mind is that the lines should be going nearly parallel to the $\mathbf{x}_1$-direction when they reach the origin, this is because the $\mathbf{x}_2$ part gets small faster. This example is very hard to draw because the two eigenvectors are so nearly in the same direction themselves, next we will look at a more typical example where they are closer to being perpendicular.

Consider the system

$$y_1' = -y_1 + \frac{1}{4}y_2$$

$$y_2' \quad = \quad -2y_2 \tag{11}$$

In matrix form this is $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} -1 & \frac{1}{4} \\ 0 & -2 \end{pmatrix} \tag{12}$$

This has eigenvalue $\lambda_1 = -1$ with eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \tag{13}$$

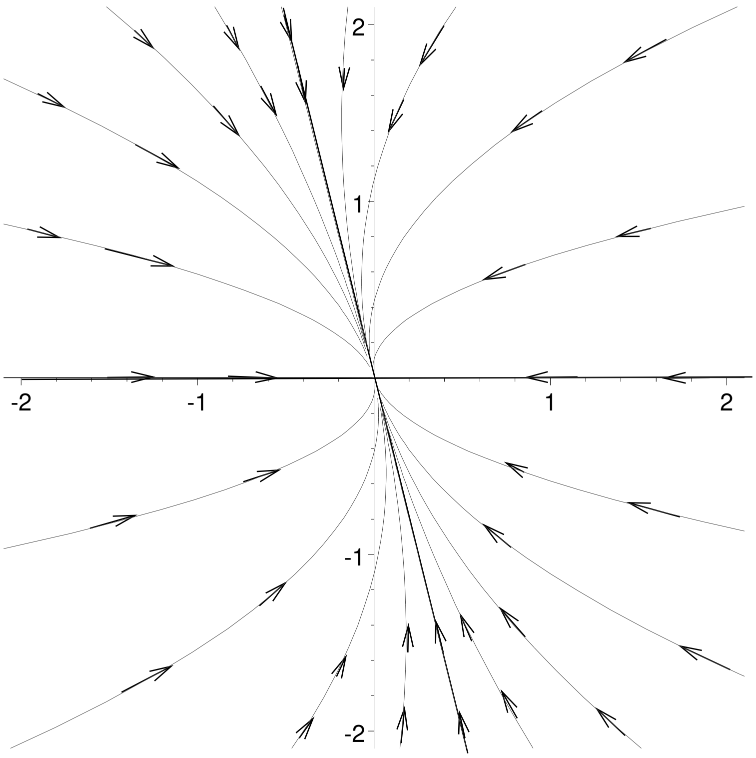
and eigenvector $\lambda_2 = -2$ with eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{14}$$

and so the solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{15}$$

Now, once again, all trajectories go inwards and they go in quicker in the $\mathbf{x}_2$-direction, this can be seen in the phase diagram:



Note that in this diagram, the y_1-axis is the $\mathbf{x}_2$-direction.