Laplace transform of a periodic function

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If $f(t)$ is a periodic function with period $a$ we have $f(t+a) = f(t)$. The Laplace transform of $f(t)$ can be found by integrating over only one period:

$$\mathcal{L}(f) = \frac{1}{1-e^{-as}} \int_0^a f(t)e^{-st}dt \quad (1)$$

This equation can be derived as follows:

$$\mathcal{L}(f) = \int_0^\infty f(t)e^{-st}dt$$
$$= \int_0^a f(t)e^{-st}dt + \int_a^{2a} f(t)e^{-st}dt + \ldots$$
$$= \sum_{n=0}^{\infty} \int_{na}^{(n+1)a} f(t)e^{-st}dt \quad (2)$$

Next, we integrate one term of the sum using the change of variable $t' = t - na$:

$$\int_{na}^{(n+1)a} f(t)e^{-st}dt = \int_0^a f(t + na)e^{-st'-nas}dt' = \int_0^a f(t')e^{-st'-nas}dt' \quad (3)$$

Hence, dropping the prime on $t$ we get

$$\mathcal{L}(f) = \left( \sum_{n=0}^{\infty} e^{-san} \right) \int_0^a f(t)e^{-st}dt \quad (4)$$

and, finally, we use

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (5)$$

to make the replacement

$$\sum_{n=0}^{\infty} e^{-san} = \frac{1}{1 - e^{-sa}} \quad (6)$$

and this proves the formula.

1Conor Houghton, houghton@maths.tcd.ie please send me any corrections.