Laplace transform of a periodic function[1]

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If $f(t)$ is a periodic function with period a we have $f(t+a) = f(t)$. The Laplace transfrom of $f(t)$ can be found by integrating over only one period:

$$\mathcal{L}(f) = \frac{1}{1-e^{-as}} \int_0^a f(t)e^{-st}dt \tag{1}$$

This equation can be derived as follows:

$$\begin{aligned}
\mathcal{L}(f) &= \int_0^\infty f(t)e^{-st}dt \\
&= \int_0^a f(t)e^{-st}dt + \int_a^{2a} f(t)e^{-st}dt + \ldots \\
&= \sum_{n=0}^\infty \int_{na}^{(n+1)a} f(t)e^{-st}dt
\end{aligned} \tag{2}$$

Next, we integrate one term of the sum using the change of variable $t' = t - na$:

$$\int_{na}^{(n+1)a} f(t)e^{-st}dt = \int_0^a f(t'+na)e^{-st'-nas}dt' = \int_0^a f(t')e^{-st'-nas}dt' \tag{3}$$

Hence, dropping the prime on t we get

$$\mathcal{L}(f) = \left(\sum_{n=0}^\infty e^{-san}\right) \int_0^a f(t)e^{-st}dt \tag{4}$$

and, finally, we use

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \tag{5}$$

to make the replacement

$$\sum_{n=0}^\infty e^{-san} = \frac{1}{1-e^{-sa}} \tag{6}$$

and this proves the formula.

[1]Conor Houghton, `houghton@maths.tcd.ie` please send me any corrections.