Laplace transform of a rectified wave[1]

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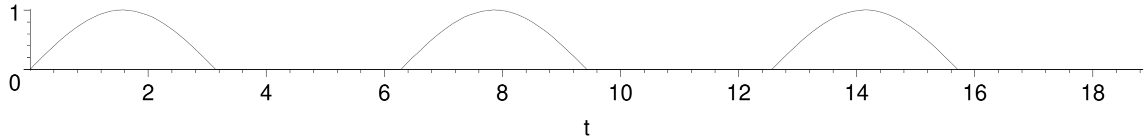
Here we will use the formala

$$\mathcal{L}(f) = \frac{1}{1-e^{-as}} \int_0^a f(t) e^{-st} dt \tag{1}$$

to find the Laplace transform of a rectified wave

$$f(t) = \begin{cases} \sin t & \sin t > 0 \\ 0 & \sin t \leq 0 \end{cases} \tag{2}$$



This is periodic with period 2π.

So we substitute this into the formula

$$\mathcal{L}(f) = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} f(t) e^{-st} dt = \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} \sin t e^{-st} dt \tag{3}$$

We need to do the integral. There are two obvious ways, the first is to split the sine into exponentials

$$\begin{aligned} \int_0^\pi \sin t e^{-st} dt &= \frac{1}{2i}\left(\int_0^\pi e^{(i-s)t} dt - \int_0^\pi e^{-(i+s)t} dt \right) \\ &= \frac{1}{2i}\left[\frac{1}{i-s}\left(e^{(i-s)\pi} - 1\right) + \frac{1}{i+s}\left(e^{-(i+s)\pi} - 1\right)\right] \end{aligned} \tag{4}$$

Now, we use

$$e^{i\pi} = e^{-i\pi} = -1 \tag{5}$$

and

$$\begin{aligned} \frac{1}{i-s} &= \frac{1}{i-s}\frac{-i-s}{-i-s} = -\frac{s+i}{s^2+1} \\ \frac{1}{i+s} &= \frac{1}{i+s}\frac{-i+s}{-i+s} = \frac{s-i}{s^2+1} \end{aligned} \tag{6}$$

to get

$$\int_0^\pi \sin t e^{-st} dt = \frac{1+e^{-s\pi}}{1+s^2} \tag{7}$$

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or

$$\mathcal{L}(f) = \frac{1}{s^2+1}\frac{1+e^{-s\pi}}{1-e^{-2s\pi}} = \frac{1}{s^2+1}\frac{1}{1-e^{-s\pi}} \tag{8}$$

where the final equality uses

$$1 - e^{-2s\pi} = \left(1 - e^{-s\pi}\right)\left(1 + e^{-s\pi}\right) \tag{9}$$

The other way to do the integral is to integrate by parts. Briefly, write

$$\begin{aligned} I = \int_0^\pi \sin t e^{-st} dt &= -\frac{1}{s}\int_0^\pi \cos t e^{-st} dt \\ &= -\frac{1}{s}\left[-\frac{1}{s}\left(e^{-\pi s}+1\right) + \frac{1}{s}I\right] \end{aligned} \tag{10}$$

and solve for I to get the answer given at (7) above.