Laplace transform of a rectified wave¹

16 November 2001

Here we will use the formala

$$\mathcal{L}(f) = \frac{1}{1 - e^{-as}} \int_0^a f(t) e^{-st} dt$$
 (1)

to find the Laplace transform of a rectified wave

$$f(t) = \begin{cases} \sin t & \sin t > 0\\ 0 & \sin t \le 0 \end{cases}$$
(2)



This is periodic with period 2π .

So we substitute this into the formula

$$\mathcal{L}(f) = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} f(t) e^{-st} dt = \frac{1}{1 - e^{-2\pi s}} \int_0^{\pi} \sin t e^{-st} dt$$
(3)

We need to do the integral. There are two obvious ways, the first is to split the sine into exponentials

$$\int_{0}^{\pi} \sin t e^{-st} dt = \frac{1}{2i} \left(\int_{0}^{\pi} e^{(i-s)t} dt - \int_{0}^{\pi} e^{-(i+s)t} dt \right)$$
$$= \frac{1}{2i} \left[\frac{1}{i-s} \left(e^{(i-s)\pi} - 1 \right) + \frac{1}{i+s} \left(e^{-(i+s)\pi} - 1 \right) \right]$$
(4)

Now, we use

$$e^{i\pi} = e^{-i\pi} = -1 \tag{5}$$

and

$$\frac{1}{i-s} = \frac{1}{i-s} \frac{-i-s}{-i-s} = -\frac{s+i}{s^2+1}$$
$$\frac{1}{i+s} = \frac{1}{i+s} \frac{-i+s}{-i+s} = \frac{s-i}{s^2+1}$$
(6)

to get

$$\int_0^\pi \sin t e^{-st} dt = \frac{1 + e^{-s\pi}}{1 + s^2} \tag{7}$$

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or

$$\mathcal{L}(f) = \frac{1}{s^2 + 1} \frac{1 + e^{-s\pi}}{1 - e^{-2s\pi}} = \frac{1}{s^2 + 1} \frac{1}{1 - e^{-s\pi}}$$
(8)

where the final equality uses

$$1 - e^{-2s\pi} = (1 - e^{-s\pi}) (1 + e^{-s\pi})$$
(9)

The other way to do the integral is to integrate by parts. Briefly, write

$$I = \int_0^{\pi} \sin t e^{-st} dt = -\frac{1}{s} \int_0^{\pi} \cos t e^{-st} dt$$
$$= -\frac{1}{s} \left[-\frac{1}{s} \left(e^{-\pi s} + 1 \right) + \frac{1}{s} I \right]$$
(10)

and solve for I to get the answer given at (7) above.