Laplace transform of a rectified wave[1]

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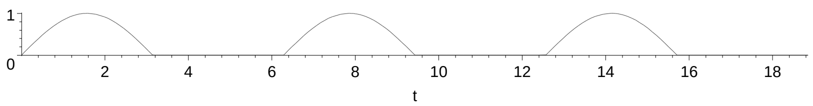
Here we will use the formala

$$\mathcal{L}(f) = \frac{1}{1-e^{-as}}\int_0^a f(t)e^{-st}dt \tag{1}$$

to find the Laplace transform of a rectified wave

$$f(t) = \begin{cases} \sin t & \sin t > 0 \\ 0 & \sin t \leq 0 \end{cases} \tag{2}$$



This is periodic with period 2π.

So we substitute this into the formula

$$\mathcal{L}(f) = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} f(t)e^{-st}dt = \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} \sin te^{-st}dt \tag{3}$$

We need to do the integral. There are two obvious ways, the first is to split the sine into exponentials

$$\begin{aligned}\int_0^\pi \sin te^{-st}dt &= \frac{1}{2i}\left(\int_0^\pi e^{(i-s)t}dt - \int_0^\pi e^{-(i+s)t}dt\right)\\ &= \frac{1}{2i}\left[\frac{1}{i-s}\left(e^{(i-s)\pi}-1\right)+\frac{1}{i+s}\left(e^{-(i+s)\pi}-1\right)\right]\end{aligned} \tag{4}$$

Now, we use

$$e^{i\pi} = e^{-i\pi} = -1 \tag{5}$$

and

$$\begin{aligned}\frac{1}{i-s} &= \frac{1}{i-s}\frac{-i-s}{-i-s} = -\frac{s+i}{s^2+1}\\ \frac{1}{i+s} &= \frac{1}{i+s}\frac{-i+s}{-i+s} = \frac{s-i}{s^2+1}\end{aligned} \tag{6}$$

to get

$$\int_0^\pi \sin te^{-st}dt = \frac{1+e^{-s\pi}}{1+s^2} \tag{7}$$

[1] Conor Houghton, `houghton@maths.tcd.ie` please send me any corrections.

or

$$\mathcal{L}(f) = \frac{1}{s^2+1}\frac{1+e^{-s\pi}}{1-e^{-2s\pi}} = \frac{1}{s^2+1}\frac{1}{1-e^{-s\pi}} \tag{8}$$

where the final equality uses

$$1-e^{-2s\pi} = \left(1-e^{-s\pi}\right)\left(1+e^{-s\pi}\right) \tag{9}$$

The other way to do the integral is to integrate by parts. Briefly, write

$$\begin{aligned}I = \int_0^\pi \sin te^{-st}dt &= -\frac{1}{s}\int_0^\pi \cos te^{-st}dt\\ &= -\frac{1}{s}\left[-\frac{1}{s}\left(e^{-\pi s}+1\right)+\frac{1}{s}I\right]\end{aligned} \tag{10}$$

and solve for I to get the answer given at (7) above.