The classification of critical points[1]

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The idea behind this note is to gather together in one place the different possible critical points. It only covers what has already been covered in the lectures.

The idea is that you have a pair of linear equations written in matrix form

$$\mathbf{y}' = A\mathbf{y} \tag{1}$$

where A is a two by two matrix with two eigenvectors. The other example, where A has only one eigenvector, we looked at separately and is found on page 167 of K. If there are two eigenvectors then there are five possibilities.

Both eigenvalues real

A proper node. $\lambda_1 = \lambda_2$. This is a trivial case, it is what happens if the matrix A is diagonal with equal entries, for example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

so the eigenvalue is $\lambda = 1$ but there are two eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3}$$

and

$$\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

so the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^t. \tag{5}$$

For any c_1 and c_2 the vector gets longer but its direction stays the same, so the phase trajectories are radial lines from the origin.

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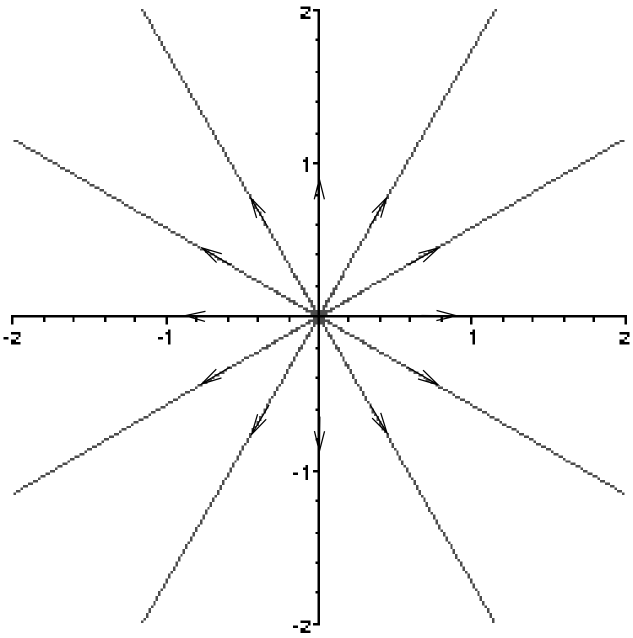


Of course, if the eigenvalue is negative the arrows point in rather than outward.

An improper node. λ_1 and λ_2 have the same sign but are different. Say they are both positive, then the general solution is given by

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t}. \tag{6}$$

Now, say $\lambda_1 > \lambda_2$ then for t very large $e^{\lambda_1 t}$ is huge compared to $e^{\lambda_2 t}$, so whatever values of c_1 and c_2 you start off with, you end up with a much bigger number multiplying $\mathbf{x}_1$ than multiplies $\mathbf{x}_2$. This means the trajectory ends up going parallel to $\mathbf{x}_1$. In the phase diagram the bend as they go away from the origin so that they become more and more parallel to $\mathbf{x}_1$.

As an example, consider the equations that came up in question 1 of problem sheet 2. This had

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \tag{7}$$

with eigenvalues and vectors $\lambda_1 = 4$ with

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8}$$

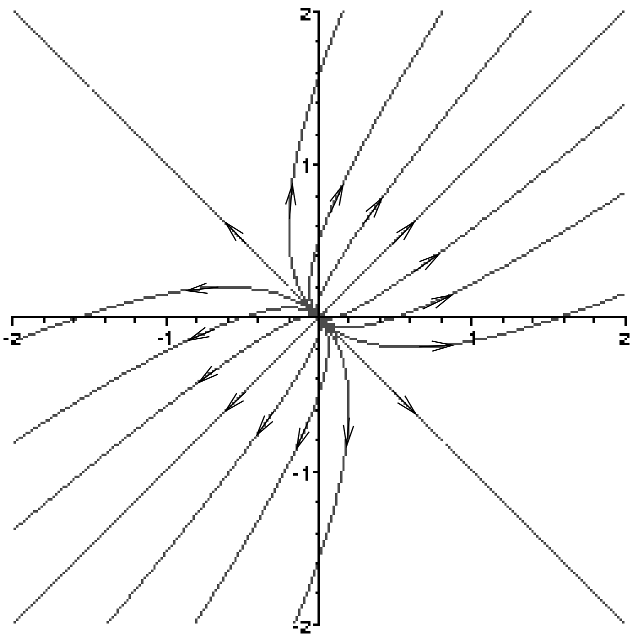
and $\lambda_2 = 2$ with

$$\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{9}$$

so the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}. \tag{10}$$

The phase-diagram is



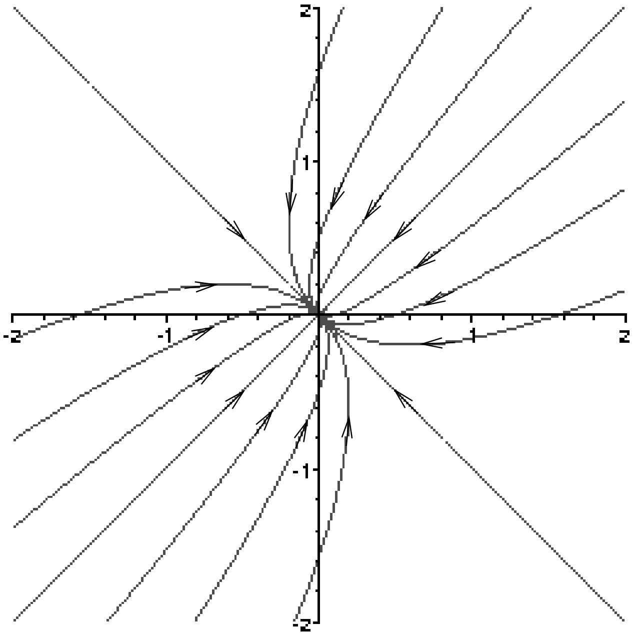
As with the proper node, it could also happen that the two eigenvalues are negative, in which case the arrows would point inward rather than outward. A long way from the origin, the trajectories bend so that they are parallel to the larger, in the sense of more negative, of the two eigenvalues. The way to think about this is to ask, for a given c_1 and c_2 where did the trajectory come from, in other words what happens for large negative t. Thus, if $\lambda_1 = -4$ with

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{11}$$

and $\lambda_2 = -2$ with

$$\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{12}$$

the phase diagram would be



A saddle-point. λ_1 and λ_2 have different signs. As before the general solution is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t}. \tag{13}$$

Say λ_1 is positive and λ_2 is negative. If you start somewhere along the $\mathbf{x}_2$ direction, then you go inwards because the number multiplying $\mathbf{x}_2$ gets smaller and smaller as time passes because of the exponential with a negative power. If you start of anywhere else, the number multiplying $\mathbf{x}_1$ gets bigger and bigger while the number multiplying $\mathbf{x}_2$ gets smaller and smaller. Thus, as you go out the trajectory gets closer and closer to $\mathbf{x}_1$. This is different from the improper node where one exponential gets much bigger than another so that the trajectory ends up parallel to one of the eigenvectors. Here one of the exponentials actually gets small, so the trajectory ends up getting closer and closer to the line along one of the eigenvectors.

An example is question 2 in problem sheet 2. Another example is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

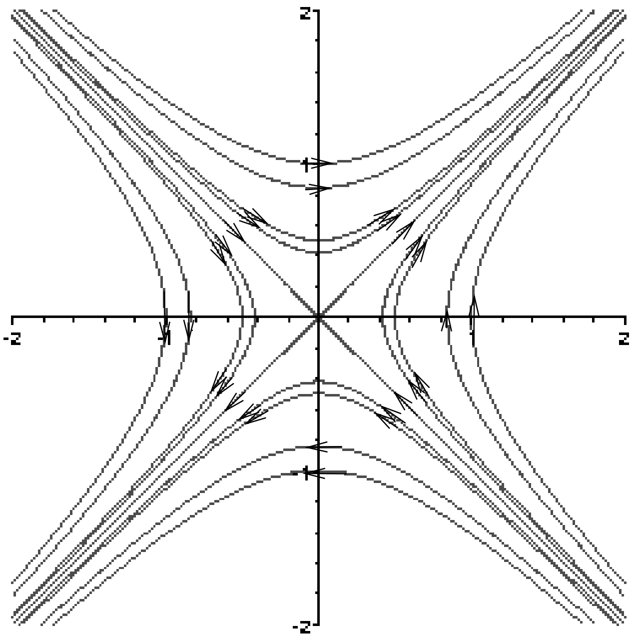
so the eigenvalue is $\lambda_1 = 1$ but there are two eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{15}$$

and $\lambda = -1$ with

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{16}$$

the phase diagram is



Eigenvalues complex

When the eigenvalues are complex they are complex conjugates of each other. There are two cases.

A center. λ_1 and λ_2 both pure imaginary. In this case the trajectories are circles or ellipses. Often the easiest thing to do to plot the trajectories is to start on the y_1 axis and use

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{17}$$

An example is

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \tag{18}$$

so the eigenvalue is $\lambda_1 = 2i$ but there are two eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix} \tag{19}$$

and $\lambda = -2i$ with

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -2i \end{pmatrix} \tag{20}$$

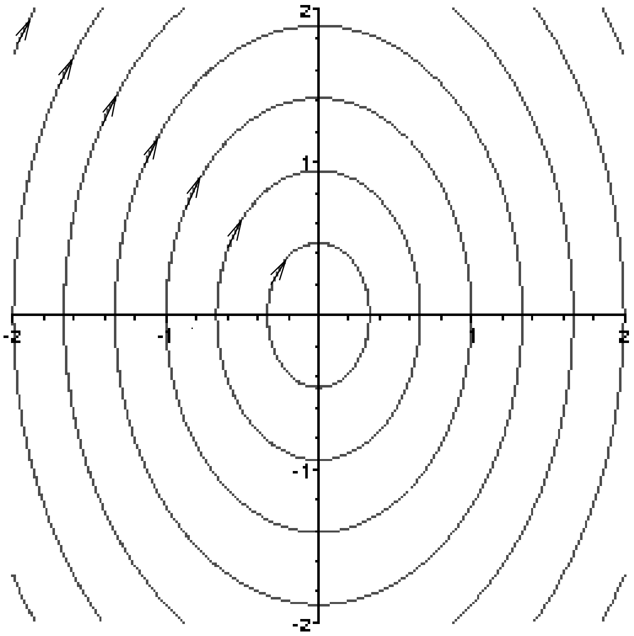
This means the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} + c_2 \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-2it} \tag{21}$$

so begin on the y_1 axis by letting $y_1(0) = r$ and $y_2(0) = 0$. The solution is then

$$\mathbf{y} = r \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} \tag{22}$$

so the trajectories are ellipsis. Every $t = \pi$ the system comes back around and the vertical radius is twice the horizontal radius. The phase diagram is



The trajectories go anticlockwise because the sine part is negative and the cosine part positive, think what happens for small t, y_1 is still pretty close to r and y_2 is negative.

A spiral. λ_1 and λ_2 have a real part. In this case the trajectories are spirals, if the real part is positive the spiral is outward, if it is negative, the spiral is inward.

An example is given in problem sheet 2, question 3. The matrix is

$$A = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \tag{23}$$

and so the spectrum is complex, $\lambda_1 = -1 + 2i$ with eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \tag{24}$$

and $\lambda_2 = -1 - 2i$ with eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix} \tag{25}$$

The solution is then

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(-1+2i)t} + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(-1-2i)t}. \tag{26}$$

Now, if $y_1(0) = 2$ and $y_2(0) = 0$, for example, this means

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \mathbf{y}(0) = c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}. \tag{27}$$

and hence $c_1 = -i$ and $c_2 = i$. Now using $\exp{(a + ib)} = \exp a \exp ib$ we have solution

$$\mathbf{y} = \left[\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{2it} + \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-2it} \right] e^{-t}. \tag{28}$$

and so

$$\begin{aligned} \mathbf{y} &= \left[\begin{pmatrix} 1 \\ -i \end{pmatrix} (\cos 2t + i \sin 2t) + \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos 2t - i \sin 2t) \right] e^{-t} && (29) \\ &= 2 \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} e^{-t} && (30) \end{aligned}$$

and this gives the inward spiral. The phase diagram is