The Laplace transform of $\cosh at \cosh bt$[1]

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This is a complicated but helpful example because it illustrates both the shift theorem and the linearity. The point is that there is no product rule for Laplace transforms, but it is possible to work out some examples. Here we work out

$$\mathcal{L}(\cosh at \cosh bt) \tag{1}$$

Start by splitting up $\cosh at$

$$\cosh at = \frac{e^{at} + e^{-at}}{2} \tag{2}$$

and so, by linearity,

$$\mathcal{L}(\cosh at \cosh bt) = \frac{1}{2}\mathcal{L}(e^{at} \cosh bt + e^{-at} \cosh bt) = \frac{1}{2}\mathcal{L}(e^{at} \cosh bt) + \frac{1}{2}\mathcal{L}(e^{at} \cosh bt) \tag{3}$$

and use the shift theorem: if $\mathcal{L}(f(t)) = F(s)$ then

$$\mathcal{L}(e^{at} f(t)) = F(s-a), \tag{4}$$

along with the known Laplace transform of $\cosh bt$

$$\mathcal{L}(\cosh bt) = \frac{s}{s^2 - b^2} \tag{5}$$

to get

$$\mathcal{L}(\cosh at \cosh bt) = \frac{1}{2}\left[\frac{s-a}{(s-a)^2 - b^2} + \frac{s+a}{(s+a)^2 - b^2}\right] \tag{6}$$

This is the answer but it is nice to follow this through and add the two fractions. Using $(s \pm a)^2 - b^2 = s^2 + a^2 - b^2 \pm 2as$ we have

$$\begin{aligned}
\frac{1}{2}\left[\frac{s-a}{(s-a)^2 - b^2} \right. & + \left. \frac{s+a}{(s+a)^2 - b^2}\right] \\
&= \frac{1}{2}\left[\frac{(s-a)(s^2 + a^2 - b^2 + 2as) + (s+a)(s^2 + a^2 - b^2 - 2as)}{(s^2 + a^2 - b^2)^2 - 4a^2 s^2}\right] \\
&= \frac{(s[(s^2 - a^2 - b^2])}{(s^2 - (a-b)^2)(s^2 - (a+b)^2)}
\end{aligned} \tag{7}$$

So, in conclusion

$$\mathcal{L}(\cosh at \cosh bt) = \frac{s[(s^2 - a^2 - b^2]}{(s^2 - (a-b)^2)(s^2 - (a+b)^2)} \tag{8}$$

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