An example of solving an inhomogeneous linear system[1]

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Consider the system

$$\begin{aligned} y_1' &= 2y_2 + t \\ y_2' &= 2y_1 + 1 \end{aligned} \tag{1}$$

with $y_1(0) = -1$ and $y_2(0) = 0$. Writing this in matrix form gives

$$\mathbf{y}' = A\mathbf{y} + \begin{pmatrix} t \\ 1 \end{pmatrix} \tag{2}$$

where

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{3}$$

A has eigenvalue $\lambda_1 = 2$ with eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4}$$

and $\lambda_2 = -2$ with eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5}$$

Please don't think all eigenvectors look like this, its seems half the time I try to think of an example I end up with the same eigenvectors, but this is just a coincidence.

Anyway, We expand the solution over the eigenvectors:

$$\mathbf{y}(t) = f_1(t)\mathbf{x}_1 + f_2(t)\mathbf{x}_2 \tag{6}$$

This is substituted into the equation. Using the fact that $A\mathbf{x}_1 = 2\mathbf{x}_1$ and $A\mathbf{x}_2 = -2\mathbf{x}_2$, this gives

$$f_1'\mathbf{x}_1 + f_2'\mathbf{x}_2 = 2f_1\mathbf{x}_1 - 2f_2\mathbf{x}_2 + \begin{pmatrix} t \\ 1 \end{pmatrix} \tag{7}$$

Now, we want seperate equations for the f_1 part and the f_2 part. To do this we must split the forcing term up between the two eigenvectors:

$$\begin{pmatrix} t \\ 1 \end{pmatrix} = g_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + g_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{8}$$

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so $g_1 + g_2 = t$ and $g_1 - g_2 = 1$. Solving the simultaneous equations gives

$$\begin{aligned} g_1 &= \frac{1+t}{2} \\ g_2 &= \frac{-1+t}{2} \end{aligned} \tag{9}$$

and so

$$f_1'\mathbf{x}_1 + f_2'\mathbf{x}_2 - 2f_1\mathbf{x}_1 + 2f_2\mathbf{x}_2 = \frac{1+t}{2}\mathbf{x}_1 + \frac{-1+t}{2}\mathbf{x}_2 \tag{10}$$

If we seperate the $\mathbf{x}_1$ equation and the $\mathbf{x}_2$ equation then

$$f_1' - 2f_1 = \frac{t+1}{2} \tag{11}$$

and

$$f_2' + 2f_2 = \frac{t-1}{2} \tag{12}$$

This is solved in the usual way

$$f_1 = C_1e^{2t} + e^{2t}\int e^{-2t}\left(\frac{t+1}{2}\right)dt \tag{13}$$

We need to integrate by parts

$$\begin{aligned} \int e^{-2t}\left(\frac{t+1}{2}\right)dt &= -\frac{1}{2}e^{-2t}\left(\frac{t+1}{2}\right) + \frac{1}{2}\int e^{-2t}\left(\frac{1}{2}\right)dt \\ &= -\frac{1}{2}e^{-2t}\left(\frac{t+1}{2}\right) - \frac{1}{8}e^{-2t} \\ &= -\frac{3}{8}e^{-2t} - \frac{t}{4}e^{-2t} \end{aligned} \tag{14}$$

Putting this back in we get

$$\begin{aligned} f_1 &= C_1e^{2t} + e^{2t}\left(-\frac{3}{8}e^{-2t} - \frac{t}{4}e^{-2t}\right) \\ &= C_1e^{2t} - \frac{3}{8} - \frac{t}{4} \end{aligned} \tag{15}$$

The same sort of calculation gives

$$f_2 = C_2e^{-2t} - \frac{3}{8} + \frac{t}{4} \tag{16}$$

Putting all this together we have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(C_1e^{2t} - \frac{3}{8} - \frac{t}{4}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(C_2e^{-2t} - \frac{3}{8} + \frac{t}{4}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{17}$$

Now, putting in the initial conditions $y_1(0) = -1$ and $y_2(0) = 0$ we get

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \left(C_1 - \frac{3}{8}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(C_2 - \frac{3}{8}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{18}$$

so

$$-1 = C_1 + C_2 - \frac{3}{4} \tag{19}$$

$$0 = C_1 - C_2 \tag{20}$$

so $C_1 = C_2 = -1/8$ and

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(-\frac{1}{8}e^{2t} - \frac{3}{8} - \frac{t}{4}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(-\frac{1}{8}e^{-2t} - \frac{3}{8} + \frac{t}{4}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{21}$$