Convolution Example[1]

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In this example we will find the f such that

$$\mathcal{L}(f) = \left(\frac{s}{s^2+1}\right)^2 \tag{1}$$

using the convolution theorem. In fact, applying the theorem is the easy bit

$$\mathcal{L}(f) = [\mathcal{L}(\cos t)]^2 = \mathcal{L}(\cos t * \cos t) \tag{2}$$

and so $f(t) = \cos t * \cos t$. Now we need to work out the convolution:

$$\cos t * \cos t = \int_0^t \cos\tau \cos(t-\tau) d\tau \tag{3}$$

We can then ue the formula

$$2\cos A \cos B = \cos(A+B) + \cos(A-B) \tag{4}$$

which can be found in the log tables. Then,

$$\cos t * \cos t = \frac{1}{2}\int_0^t \cos t d\tau + \frac{1}{2}\int_0^t \cos(2\tau - t) d\tau \tag{5}$$

and then using the formula

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \tag{6}$$

which is also in the log tables, we find

$$\begin{aligned}
\cos t * \cos t &= \frac{1}{2}t\cos t + \frac{1}{2}\int_0^t (\cos 2\tau \cos t + \sin 2\tau \sin t) d\tau \\
&= \frac{1}{2}t\cos t + \frac{1}{4}\cos t \sin 2t - \frac{1}{4}\sin t \cos 2t + \frac{1}{4}\sin 2t \\
&= \frac{1}{2}t\cos t + \frac{1}{4}\sin t + \frac{1}{4}\sin 2t \\
&= \frac{1}{2}t\cos t + \frac{1}{2}\sin t
\end{aligned} \tag{7}$$

where I have used the formula $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

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