Example of solving a linear second order differential equation[1]

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Consider the second order equation:

$$y'' + 3y' + 2y = 0 \tag{1}$$

with initial conditions $y(0) = 2$ and $y'(0) = -1$. We start by converting this into two first order equations. First, let $y_1 = y$ and $y_2 = y'$. This means that $y_2' = y''$ and using the equation itself this gives $4y_2' = -3y' - 2y = -3y_2 - 2y_1$. Hence,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -2y_1 - 3y_2 \end{aligned} \tag{2}$$

Now we write this in matrix form:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{3}$$

Now, we can work out the eigenvalues and eigenvectors for the matrix, we find $\lambda_1 = -2$ with

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{4}$$

and $\lambda_2 = -1$ with

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5}$$

By the usual reasoning the solution is now

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6}$$

Next, the initial conditions. In terms of $y_1 = y$ and $y_2 = y'$ we have $y_1(0) = 2$ and $y_1(0) = -1$ so, putting $t = 0$ into the solution, this gives:

$$\begin{aligned} 2 &= C_1 + C_2 \\ -1 &= -2C_1 - C_2 \end{aligned} \tag{7}$$

These can be solved to give $C_1 = -1$ and $C_2 = 3$. Hence,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 3e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{8}$$

or, using $y_1 = y$,

$$y = -e^{-2t} + 3e^{-t} \tag{9}$$

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