Solving and ordinary inhomogeneous equation. [1]

10 February 2003

An ordinary inhomogeneous equation is an equation of the form

$$y' + ry = f(t) \tag{1}$$

where r is some constant and $f(t)$ is some function. An example would be

$$y' + 3y = e^{4t} \tag{2}$$

The way to solve one of these is to use an integrating factor. Lets use an integrating factor to solve the general example (1) and this will give a general formula. The crucial point is that you can turn the first two terms into a single term by multiplying across by an exponential. Multiplying both sides by e^{rt} we get

$$e^{rt}y' + e^{rt}ry = e^{rt}f(t) \tag{3}$$

but, using the product rule,

$$\left(e^{rt}y\right)' = \left(e^{rt}\right)' y + e^{rt}y' = re^{rt}y + e^{rt}y' \tag{4}$$

Hence, the equation is

$$\left(e^{rt}y\right)' = e^{rt}f(t) \tag{5}$$

or

$$\frac{d\left(e^{rt}y\right)}{dt} = e^{rt}f(t) \tag{6}$$

so

$$\int d\left(e^{rt}y\right) = \int e^{rt}f(t)dt \tag{7}$$

and doing the integral on the left, $\int dx = x + C$, gives

$$e^{rt}y = C + \int e^{rt}f(t)dt \tag{8}$$

where, for convenience, I have put the constant on the right. Finally, multiplying across by the e^{-rt} we get the formula

$$y = Ce^{-rt} + e^{-rt}\int e^{rt}f(t)dt \tag{9}$$

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and we should bear in mind that the integral on the right also gives an arbitrary constant, but it can automatically be absorbed into the arbitrary constant, C, that is already there. Of course, if there is some initial condition then this constant is fixed by the initial condition.

Now, if you are given an example to do, such as the

$$y' + 3y = e^{4t} \tag{10}$$

above, either you can following along the arguement above using an integrating factor, or, you can decide what r and f are and put them into the general formula. Here $r = 3$ and $f = e^{4t}$ so we get

$$\begin{aligned} y &= Ce^{-3t} + e^{-3t}\int e^{3t}e^{4t}dt = Ce^{-3t} + e^{-3t}\int e^{7t}dt \\ &= Ce^{-3t} + e^{-3t}\frac{1}{7}e^{7t} = Ce^{-3t} + \frac{1}{7}e^{4t} \end{aligned} \tag{11}$$

and that is the answer.

By the way, we don't do it in this course, but you can use the integrating factor method even if r isn't a constant. It becomes harder to choose the integrating factor, but the idea is basically the same.