2010/11 take home exam, two hour exam, do four questions.

1. (a) Prove

$$\int_{-\pi}^{\pi} e^{int}e^{-imt}dt = 2\pi\delta_{mn}$$

   (b) How is this identities used to derive the formula for the c_n in the complex Fourier series.

   (c) Find the complex Fourier series of the function $f(t)$ with period 2π defined by

$$f(t) = \begin{cases} 0 & -\pi < t < -a \\ 1 & -a < t < a \\ 0 & a < t < \pi \end{cases}$$

   where $0 < a < \pi$ is a constant.

2. Consider the Fourier expansion of $f(t) = t$ for $-\pi < t < \pi$ with $f(t + 2\pi) = f(t)$ and use the result to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3. Write the on-off pulse

$$f(t) = \begin{cases} 1 & -\pi < t < 0 \\ -1 & 0 < t < \pi \\ 0 & \text{otherwise} \end{cases}$$

   as a Fourier integral.

4. For the Gaussian curve

$$f(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

   find the Fourier transform $\widetilde{f(k)}$. You may use the integral

$$\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$$

   What is the Fourier integral form of the Dirac delta function $\delta(t)$?

5. (a) Find the general solution to

$$\ddot{y} + 4\dot{y} + 3y = 0.$$

What is the solution if $y(0) = \dot{y}(0) = 1$.

(b) Find the general solution to

$$\ddot{y} + 4\dot{y} + 3y = e^t.$$

What is the solution if $y(0) = \dot{y}(0) = 1$.

6. Using the substitution $t = e^z$ or otherwise, find the general solution of Euler equation

$$t^2\ddot{y} - 3t\dot{y} + 3y = t^3$$

Some useful formula

- A function with period L has the Fourier series expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{L}\right).$$

where

$$\begin{aligned} a_0 &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)dt \\ a_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\cos\left(\frac{2\pi nt}{L}\right) dt \\ b_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\sin\left(\frac{2\pi nt}{L}\right) dt \end{aligned}$$

- A function with period L has the Fourier series expansion

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\frac{2i\pi nt}{L}.$$

where

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\exp\left(\frac{-2i\pi nt}{L}\right)dt$$

- The Fourier integral or Fourier transform:

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikt} \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)e^{-ikt} \end{aligned}$$