2009/10 supplemental paper, all questions, draft version.

1. (a) The Fourier series relies on the orthogonality of the sine and cosine functions: as an example, show

$$\int_{-\pi}^{\pi} \sin nt \sin mt dt = \pi\delta_{nm}$$

and

$$\int_{-\pi}^{\pi} \cos nt \cos mt dt = \pi\delta_{nm}$$

for integers n and m.

(b) In a similar way it can be shown that

$$\int_{-\pi}^{\pi} \sin nt \cos mt dt = 0$$

How are these identities used to derive formulas for the a_0, a_n and b_n in the Fourier series.

(c) Find the Fourier series of the function $f(t)$ with period 2π defined by

$$f(t) = \begin{cases} -1 & -\pi \geq t < 0 \\ 1 & 0 \leq t < \pi \end{cases}$$

2. Find both the Fourier series of the rectified sine wave

$$f(t) = \max(\sin t, 0) = \begin{cases} \sin t & \sin t > 0 \\ 0 & \sin t \leq 0 \end{cases} \tag{1}$$

using the complex expansion

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

If $f(t)$ is real what constraint is satisfied by the c_n. Verify this in this case.

3. What are

$$\int_{-\infty}^{\infty} \delta(t)dt \quad \int_{-\infty}^{\infty} \delta(t-1)dt \quad \int_{0}^{\infty} \delta(t-1)dt \quad \int_{-\infty}^{0} \delta(t-1)dt$$

Write

$$f(t) = \begin{cases} \sin t & |t| < \pi \\ 0 & |t| > \pi \end{cases}$$

as a Fourier integral.

4. Find the general solution of

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = e^t$$

5. Using the substitution $t = e^z$ or otherwise, find the general solution of Euler equation

$$t^2\ddot{y} + 3t\dot{y} + y = t^2$$

6. Assuming the solution of

$$(1-t)\dot{y} + y = 0 \tag{2}$$

has a series expansion about $t = 0$ work out the recursion relation. Write out the first few terms. What is the solution with $y(0) = 2$.

Some useful formula

- A function with period L has the Fourier series expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{L}\right).$$

where

$$\begin{aligned} a_0 &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)dt \\ a_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\cos\left(\frac{2\pi nt}{L}\right)dt \\ b_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\sin\left(\frac{2\pi nt}{L}\right)dt \end{aligned}$$

- A function with period L has the Fourier series expansion

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\frac{2i\pi nt}{L}.$$

where

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\exp\left(\frac{-2i\pi nt}{L}\right)dt$$

- The Fourier integral or Fourier transform:

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikt} \\ \widetilde{f(k)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, f(t) e^{-ikt} \end{aligned}$$