2009/10 Schol paper 2, my questions, draft version.

This was the second schol paper, for students only doing one maths module.

1. Find the Fourier series for $f(t)$ where $f(t) = |t|$ when $-\pi < t < \pi$ and $f(t+2\pi) = f(t)$.

   *Solution:*So
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt \tag{1}$$
   giving, in this case
$$c_n = -\frac{1}{2\pi}\int_{-\pi}^{0} te^{-int}dt + \frac{1}{2\pi}\int_{0}^{\pi} te^{-int}dt \tag{2}$$
   So, if we change t to $-t$ in the first term dt goes to $-dt$ and there is anther change of sign from the integration limits, giving, altogether
$$c_n = -\frac{1}{2\pi}\int_{0}^{\pi} t\left(e^{int} - e^{-int}\right)dt = -\frac{1}{\pi}\int_{0}^{\pi} t\cos ntdt \tag{3}$$
   Now for $n \neq 0$ this integral can be done by parts
$$\begin{aligned}\int_0^\pi t\cos ntdt &= \left.\frac{1}{n}t\sin nt\right]_0^\pi - \frac{1}{n}\int_0^\pi \sin ntdt \\ &= \frac{1}{n^2}(\cos n\pi - 1)\end{aligned} \tag{4}$$
   and $\cos n\pi - 1$ is equal zero is n is even and -2 if n is odd, so
$$c_n = \frac{2}{\pi n^2} \tag{5}$$
   for n odd and zero otherwise, unless $n =$, then we get
$$c_0 = -\frac{1}{\pi}\int_0^\pi tdt = \frac{\pi}{2} \tag{6}$$

2. The *convolution of two functions* $f(t)$ and $g(t)$ is
$$f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

Show that

$$\mathcal{F}(f * g) = 2\pi\mathcal{F}(f)\mathcal{F}(g)$$

Solution: Here you follow the question along

$$\mathcal{F}(f * g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f * g e^{-ikt} dt \tag{7}$$

and substituting in for the definition of the convolution

$$\mathcal{F}(f * g) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau e^{-ikt} dt \tag{8}$$

The trick is to let

$$e^{-ikt} = e^{-ik(t-\tau)}e^{-it\tau} \tag{9}$$

and letting $s = t - \tau$

$$\begin{aligned}\mathcal{F}(f * g) &= 2\pi\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-ik\tau}d\tau\frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)e^{-iks}ds \\ &= \mathcal{F}(f)\mathcal{F}(g)\end{aligned} \tag{10}$$

3. Find the general solution of

$$\ddot{y} + 2\dot{y} + 2y = e^t$$

and write it in an explicitly real form.

Solution: First solve the homogeneous problem, substitute in

$$y = e^{\lambda t} \tag{11}$$

to get

$$\lambda^2 + 2\lambda + 2 = 0 \tag{12}$$

so

$$\lambda = -1 \pm i \tag{13}$$

This means the homogeneous solution is

$$y = (C_1 e^{it} + C_2 e^{-it})e^{-t} \tag{14}$$

and expanding out the complex exponentials using

$$e^{it} = \cos t + i\sin t \tag{15}$$

This is equivalent to

$$y = (A\cos t + B\sin t)e^{-t} \tag{16}$$

To find a particular solution substitute

$$y = Ce^t \tag{17}$$

to find $C = 1/5$ and hence the general solution is

$$y = (A\cos t + B\sin t)e^{-t} + \frac{1}{5}e^t \tag{18}$$

4. Using the method of Fröbenius, or otherwise, find the general solution to

$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - \alpha ty = 0$$

where $\alpha > 0$.

*Solution:*So, we substitute

$$y = \sum_{s=0}^{\infty} a_s t^{s+r} \tag{19}$$

where r is a number to be determined by the indicial equation. Now,

$$ty = \sum_{s=0}^{\infty} a_s t^{s+r+1} \tag{20}$$

whereas

$$\frac{dy}{dt} = \sum_{s=0}^{\infty} a_s(s+r)t^{s+r-1} \tag{21}$$

and

$$t\frac{d^2y}{dt^2} = \sum_{s=0}^{\infty} a_s(s+r)(s+r-1)t^{s+r-1} \tag{22}$$

so the first two terms each have to have a change of index to raise their indices by two

$$\frac{dy}{dt} = ra_0t^{s+r-1} + (r+1)a_1t^{s+r} + \sum_{s=0}^{\infty} a_{s+2}(s+r+2)t^{s+r+1} \tag{23}$$

and

$$
\begin{aligned}
t\frac{d^2y}{dt^2} &= r(r-1)a_0t^{s+r-1} \\
&+ (r+1)ra_1t^{s+r} + \sum_{s=0}^{\infty} a_{s+2}(s+r+2)(s+r+1)t^{s+r+1} \qquad (24)
\end{aligned}
$$

Next we need to substitute everything to get the full equation for the components; for simplicity lets do the two separate terms separately, so the t to the power of $s+r-1$ gives

$$r(r-1)a_0 + 2ra_0 = 0 \qquad (25)$$

so, if a_0 is not zero this means $r = 0$ or $r = -1$. Next the $s+r$ power gives

$$r(r+1)a_1 + 2(r+1)a_1 = 0 \qquad (26)$$

Finally, the sum, the general power of t gives

$$[(s+r+2)(s+r+1) + 2(s+r+2)]a_{s+2} - \alpha a_s = 0 \qquad (27)$$

and hence

$$a_{s+2} = \frac{\alpha a_s}{(s+r+2)(s+r+3)} \qquad (28)$$

with $a+0$ arbitrary, $a_1 = 0$ and $r = 0$ or $r = -1$.