2009/10 Schol paper 1, my questions, outline solutions.

1. The Laplace transform of a function $f(t)$, $t \geq t$ is a function of s defined as
$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$
Show
$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$
where a is a constant and you may assume $s > a$. Show that,
$$\mathcal{L}\left[\frac{df}{dt}\right] = s\mathcal{L}[f(t)] - f(0)$$
where you can assume
$$\lim_{t\to\infty} f(t)e^{-st} = 0$$
By taking the Laplace transform of both sides of the equation solve
$$\frac{df}{dt} = 2f$$
where $f(0) = -1$.

Solution: So you just have to follow this question along; first
$$\mathcal{L}[e^{at}] = \int_0^\infty e^{(a-s)t}dt = \frac{1}{a-s}\left. e^{(a-s)t}\right]_0^\infty = \frac{1}{s-a} \tag{1}$$
where we have assumed that $s > a$ in order to evaluate the exponential at infinity.

Next, from the definition
$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}e^{st}dt \tag{2}$$
Now use integration by parts
$$\begin{aligned}\mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^\infty \frac{df}{dt}e^{-st}dt \\ &= \left. f(t)e^{-st}\right]_0^\infty - s\int_0^\infty \frac{df}{dt}e^{-st}dt = s\mathcal{L}[f(t)] - f(0) \end{aligned} \tag{3}$$

Finally taking the Laplace transform of both sides of the differential equation gives
$$s\mathcal{L}[f(t)] + 1 = 2\mathcal{L}[f(t)] \tag{4}$$
so, solving gives
$$\mathcal{L}[f(t)] + 1 = -\frac{1}{s-2} \tag{5}$$
and, hence, by the above, and noting that the transform is linear
$$f(t) = -e^{2t} \tag{6}$$

2. Write the on-off pulse
$$f(t) = \begin{cases} 1 & -\pi < t < 0 \\ -1 & 0 < t < \pi \\ 0 & \text{otherwise} \end{cases}$$
as a Fourier integral.

Solution: Well lets apply the definition and integrate
$$\begin{aligned}\widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^\infty f(t)e^{-ikt}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^0 e^{-ikt}dt - \frac{1}{2\pi}\int_0^\pi e^{-ikt}dt \\ &= \frac{1}{2\pi}\frac{1}{-ik}\left. e^{-ikt}\right]_{-\pi}^0 - \frac{1}{2\pi}\frac{1}{-ik}\left. e^{-ikt}\right]_0^\pi \\ &= \frac{i}{2\pi k}\left(1 - e^{ik\pi}\right) - \frac{i}{2\pi k}\left(e^{-ik\pi} - 1\right)\end{aligned} \tag{7}$$
Now, putting all that together
$$\widetilde{f(k)} = \frac{i}{\pi k} - \frac{i}{\pi k}\cos \pi k \tag{8}$$

3. Find the general solution of
$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = e^t$$

Solution: First off, substitute into the homogenous equation

$$y = e^{\lambda t} \tag{9}$$

giving

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \tag{10}$$

which factorizes as

$$(\lambda - 1)^3 = 0 \tag{11}$$

so the homogenous equation has general solution

$$y = (C_1 + tC_2 + t^2 C_3)e^t \tag{12}$$

Hence, to match the right hand side we need to substitute

$$y = Ct^3 e^t \tag{13}$$

so

$$\begin{aligned} y' &= Ct^3e^t + 3Ct^2e^t \\ y'' &= Ct^3e^t + 6Ct^2e^t + 6Cte^t \\ y''' &= Ct^3e^t + 9Ct^2e^t + 18Cte^t + 6Ce^t \end{aligned} \tag{14}$$

All the terms with t outside the exponential in them cancel and we get

$$6C = 1 \tag{15}$$

and hence the solution to the inhomogenous equation is

$$y = \left(C_1 + tC_2 + t^2C_3 + \frac{1}{6}t^3\right)e^t \tag{16}$$

4. Find the general solution of

$$t^2\ddot{y} + 3t\dot{y} + y = t^2$$

Solution: So this is an Euler equation, let

$$t = e^z \tag{17}$$

so

$$\frac{dy}{dt} = \frac{dy}{dz}\frac{dz}{dt} = \frac{1}{t}\frac{dy}{dz} \tag{18}$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{1}{t}\frac{dy}{dz}\right) = -\frac{1}{t^2}\frac{dy}{dz} + \frac{1}{t^2}\frac{d^2y}{dz^2} \tag{19}$$

Now, the equation becomes

$$\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + y = e^{2z} \tag{20}$$

which can be solved using the normal substitution to give

$$y = (C_1 + C_2 z)e^{-z} + \frac{1}{9}e^{2z} \tag{21}$$

or

$$y = (C_1 + C_2 \log t)\frac{1}{t} + \frac{1}{9}t^2 \tag{22}$$