2009/10 Schol paper 1, all questions, draft version.

The first four questions are Derek Kitson's, the best six answers went to the mark with no restriction on how this was divided between Derek and me. The standard Fourier formula sheet was also included, see the Schol paper 2 for example.

1. Let C be the plane curve

$$y = \ln x, \quad x > 0$$

   (a) Find the curvature of C.

   (b) Describe the behaviour of the curve as $x \to \infty$.

   (c) Find the point on C at which the curve has maximum curvature.

2. Find the minimum volume of a solid region in the first octant in $\mathbf{R}^3$ which is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and by a tangent plane to the sphere

$$x^2 + y^2 + z^2 = 1$$

3. (a) Compute the line integral

$$\int_C e^{yz}\,dx + (xz\,e^{yz} + z\cos y)\,dy + (xy\,e^{yz} + \sin y)\,dz$$

   where C is the curve consisting of the line segment joining $(1, 0, 1)$ to $(0, 0, 0)$ followed by the parabola

$$x = t, \quad y = t^2, \quad z = 0, \quad 0 \le t \le 1$$

   (b) Determine if the vector field

$$\mathbf{F}(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

   is conservative on its domain.

4. Compute the integral
$$\iiint_E (6+4y)\,dV$$
where E is the solid region in $\mathbf{R}^3$ bounded above by the ellipsoid
$$\frac{x^2}{9}+\frac{y^2}{4}+z^2=1$$
and bounded below by the plane $z=0$.

5. The Laplace transform of a function $f(t)$, $t \geq t$ is a function of s defined as
$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$
Show
$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$
where a is a constant and you may assume $s > a$. Show that,
$$\mathcal{L}\left[\frac{df}{dt}\right] = s\mathcal{L}[f(t)] - f(0)$$
where you can assume
$$\lim_{t\to\infty} f(t)e^{-st} = 0$$
By taking the Laplace transform of both sides of the equation solve
$$\frac{df}{dt} = 2f$$
where $f(0) = -1$.

6. Write the on-off pulse
$$f(t) = \begin{cases} 1 & -\pi < t < 0 \\ -1 & 0 < t < \pi \\ 0 & \text{otherwise} \end{cases}$$
as a Fourier integral.

7. Find the general solution of
$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 1 = e^t$$

8. Find the general solution of
$$t^2\ddot{y} + 3t\dot{y} + y = t^2$$