2009/10 Schol paper 2, my questions, draft version.

This was the second schol paper, for students only doing one maths module.

1. Find the Fourier series for $f(t)$ where $f(t) = |t|$ when $-\pi < t < \pi$ and $f(t + 2\pi) = f(t)$.
2. The *convolution of two functions* $f(t)$ and $g(t)$ is
$$f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$
Show that
$$\mathcal{F}(f * g) = 2\pi\mathcal{F}(f)\mathcal{F}(g)$$
3. Find the general solution of
$$\ddot{y} + 2\dot{y} + 2 = e^t$$
and write it in an explicitly real form.
4. Using the method of Fröbenius, or otherwise, find the general solution to
$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - \alpha t y = 0$$
where $\alpha > 0$.

Some useful formula

- A function with period L has the Fourier series expansion
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{L}\right).$$
where
$$\begin{aligned} a_0 &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)dt \\ a_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\cos\left(\frac{2\pi nt}{L}\right)dt \\ b_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\sin\left(\frac{2\pi nt}{L}\right)dt \end{aligned}$$

- A function with period L has the Fourier series expansion

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp \frac{2i\pi nt}{L}.$$

where

$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} f(t) \exp \left(\frac{-2i\pi nt}{L} \right) dt$$

- The Fourier integral or Fourier transform:

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikt} \\ \widetilde{f(k)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, f(t) e^{-ikt} \end{aligned}$$